# Preliminaria

DEFINITIONS: A **Banach algebra** is a pair $(A, \|\cdot\|)$, where $A$ is an algebra (in this course we restrict ourselves to algebras over $\mathbb{C}$) with a norm $\|\cdot\|$ making $A$ a complete space and satisfying $\|x\cdot y\| \leq \|x\|\|y\|$ for any $x, y \in A$. In most cases we assume that $A$ is a **unital** algebra, i.e. it possesses a unit $\mathbf{1}$. Then by $G_A$ we denote the **group of invertible elements** of $A$. The **spectrum** of an element $x \in A$ is defined as $\sigma(x) := \{\lambda \in \mathbb{C} : (\lambda\mathbf{1} - x) \notin G_A\}$. We also consider
$\mathrm{Sp}(A) := \{\omega : A \to \mathbb{C} : \omega$ is a non-zero homomorphism of algebras $\}$, called the **spectrum of** $A$.

We say that the mapping $* : A \to A$ is an **involution**, if it is conjugate-linear (i.e. additive and satisfying $(\lambda x)^* = \bar{\lambda}x^*$ and such that $(x^*)^* = x, (xy)^* = y^*x^*$ for all $x, y \in A$. Note that if $1 \in A$, then $1 = 1^{**} = (1\mathbf{1}^*)^* = 1^{**}1^* = 1^*$ and $(x^{-1})^* = (x^*)^{-1}$. Hence $\sigma(x^*) = \{\bar{\lambda} : \lambda \in \sigma(x)\}$. If additionally

$$\forall_{x\in A}\ \|x\|^2 = \|x^*x\| \qquad (C*),$$

we say that $A$ is a **C*-algebra**[1].

EXAMPLES: Two most important examples of unital C*-algebras are $C(K)$ and $\mathcal{B}(H)$.

Here $C(K)$ is the commutative algebra of continuous functions on a compact Hausdorff space $K$ with sup-norm, point-wise algebraic operations and involution given by complex conjugation.

A non-commutative C*-algebra is the space $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space $H$.

The easiest example of a commutative, non-unital C-* algebra is the following one: If $\Omega$ is locally compact, but non-compact, then $C_0(\Omega) = \{f \in C(\Omega) : \forall_{\epsilon>0}\exists_{K\subset\subset\Omega}\forall_{t\in\Omega\setminus K}|f(t)| < \epsilon\}$ is the algebra of continuous functions vanishing at infinity. Here $K \subset\subset \Omega$ means that $K$ is a compact subset of $\Omega$. By taking a 1-point (Alexandroff) compactification $\hat{\Omega} := \Omega \cup \{\infty\}$, where the set of all open neighbourhoods of $\infty$ is the family $\{\hat{\Omega}\setminus K : K \subset\subset \Omega\}$. Now we identify $C_0(\Omega)$ with the maximal ideal $\{f \in C(\hat{\Omega}) : f(\infty) = 0\}$. Here we have a concrete example of a general method of adjoining the unit (here denoted by $\mathbf{1}$, a process called unitisation) of a non-unital Banach algebra $A$.

Namely, define $A_1$ as the Cartesian product $A \times \mathbb{C}$ with coordinate-wise vector operations and involution, while the product is $(x, \lambda)\cdot(y, \mu) := (xy + \lambda y + \mu x, \lambda\mu)$. If $x \in A, \lambda \in \mathbb{C}$, we write $(x, \lambda) = x + \lambda\mathbf{1}$. Its norm is defined by $\|(x, \lambda)\|_1 = \|\lambda\mathbf{1} + x\|_1 := \sup_{\|y\|\leq 1}\|xy + \lambda y\|$. It is sub-multiplicative and satisfies the C*-condition if so does $(A, \|\cdot\|)$. The unit in $A_1$ is $\mathbf{1} := (0, 1)$, as easy computations[2] show.

We can find also partial substitutes to a unit in any C*-algebra called approximate units. These are sequences or nets $(k_n)$ such that $\|x - k_nx\| \to 0$ as $n \to \infty$ (-for left approximate units).

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The left- multiplication operators $L_a : A \ni x \mapsto ax \in A$ are linear, bounded with operator norm $\|L_a\| = \|a\|$ (here we consider unital algebras), therefore if in $A$ a series $\sum_{n=0}^{\infty} x_n$ norm-converges, then so does $\sum L_a(x_n)$ and its sum equals $a \cdot \sum x_n$. The same holds true for right multiplications : $x \mapsto xa$. Now if $q := \|\mathbf{1} - x\| < 1$, applying the above with $a := \mathbf{1} - x$, $x_n := a^n$ we obtain that for $S := \sum_{n=0}^{\infty}(\mathbf{1} - x)^n$ we have $(\mathbf{1} - x)S = S - \mathbf{1}$, implying $xS = \mathbf{1}$. Similarly we deduce that $Sx = \mathbf{1}$.

**COROLLARIES, Lemmas**:

(1) For $x \in A$ if $\|\mathbf{1} - x\| < 1$, then the C.Neumann series $\sum_{n=0}^{\infty}(\mathbf{1} - x)^n$ converges to $x^{-1}$, so $x \in G_A$.

(2) The set $G_A$ of invertible elements is open.

(3) Spectra $\sigma(a)$ are compact, $r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\} \leq \|a\|$ for all $a \in A$. For $|\lambda| > \|a\|$

$$(\lambda\mathbf{1} - a)^{-1} = \tfrac{1}{\lambda}(\mathbf{1} - \tfrac{1}{\lambda}a)^{-1} = \sum_{n=0}^{\infty}\tfrac{1}{\lambda^{n+1}}a^n \qquad (*)$$

(4) The inversion operation: $G_A \ni z \mapsto z^{-1}$ is continuous.

(5) Homomorphisms $\omega \in \mathrm{Sp}(A)$ are continuous, $\omega(a) \in \sigma(a)$, $|\omega(a)| \leq \|a\|$ for $a \in A$. Even $\|\omega\| = 1$.

(6) The resolvent mapping $\mathbb{C}\setminus\sigma(a) \ni \lambda \mapsto R_\lambda := (\lambda\mathbf{1} - a)^{-1} \in A$ satisfies the Hilbert equation

$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu, \quad \text{if} \quad \lambda, \mu \in \mathbb{C}\setminus\sigma(\mathrm{a})$$

and are analytic, which means $\forall\psi \in A^*$ the mappings $\lambda \mapsto f(\lambda) := \psi(R_\lambda) \in \mathbb{C}$ are analytic on $\mathbb{C}\setminus\sigma(a)$.

(7) Spectra of elements of $A$ are non-empty.

(7a) If $ab = ba$ and $a \notin G_A$ then $ab \notin G_A$. If $\forall_{j,k}a_ja_k = a_ka_j$, then $\prod_{j=1}^{n} a_j \in G_A \leftrightarrow \forall_j a_j \in G_A$.

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[1] Combining (C*) with $\|x^*x\| \leq \|x^*\|\|x\|$, then dividing both sides of the resulting inequality by $\|x\|$ we obtain $\|x\| \leq \|x^*\|$ (and "$\geq$", by replacing $x$ with $x^*$). If $\forall_x \|x^*\| \leq \|x\|$, then inequality "$\leq$" in (C*) implies "$=$" in (C*).

[2] For any $0 < t < 1$ we find $y$ with $\|y\| = 1, t\|(x, \lambda)\|_1 \leq \|xy+\lambda y\|$. Hence $t^2\|(x, \lambda)\|_1 \leq \|xy+\lambda y\|^2 = \|(xy+\lambda y)^*(xy+\lambda y)\| = \|y^*(\lambda\mathbf{1}+x)^*(\lambda\mathbf{1}+x)y\|_1 \leq \|(\lambda\mathbf{1}+x)^*(\lambda\mathbf{1}+x)\|_1$. As $t \to 1^-$, we obtain $\|(\lambda\mathbf{1} + x)\|_1{}^2 \leq \|(\lambda\mathbf{1}+x)^*\|_1\|(\lambda\mathbf{1}+x)\|_1$. Dividing both sides by $\|(\lambda\mathbf{1} + x)\|_1$ and next using the previous footnote's conclusion we obtain (C*) for $A_1$.

(8) Let $a \in A$ and let $F(z) = \sum_{n=0}^{\infty} c_n z^n$ be holomorphic in $\mathbb{D}_\rho := \{z \in \mathbb{C} : |z| < \rho\}$ for some $\rho > r = \|a\|$. Then the series $F(a) := \sum_{n=0}^{\infty} c_n a^n$ converges and $F(\lambda) \in \sigma(F(a))$ for any $\lambda \in \sigma(a)$.
For our purpose 2 easier than (8) results will suffice:
(8a) If $p$ is either a polynomial, or the function $z \mapsto z^{-1}$, then $\sigma(p(a)) = p(\sigma(a))$.
(9) Spectral radius $r(a)$ defined in (3) above satisfies the relations

$$r(a) = \lim_{n\to\infty} (\|a^n\|)^{\frac{1}{n}} .$$

(10) This formula implies that normal elements in C*-algebras satisfy $\|x\| = r(x)$.
(11) In commutative C*-algebras if $\omega \in \mathrm{Sp}(A)$ and $x \in A$, then we have $\omega(x^*) = \overline{\omega(x)}$.
**Theorem 1** (Gelfand- Mazur) *If $A \setminus \{0\} = G_A$ (i.e. if $A$ is a field), then $A = \{\lambda\mathbf{1} : \lambda \in \mathbb{C}\}$.*
**Corollary 1.1** (Gelfand) *For commutative Banach algebras the assignment: $Sp(A) \ni \omega \mapsto \ker(\omega)$ is a bijection[3] onto the set of all maximal ideals of $A$. Moreover $\sigma(a) = \{\omega(a) : \omega \in \mathrm{Sp}(A)\}$.*

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For a commutative Banach algebra $A$ we have the Gelfand transform $\Gamma : A \ni a \mapsto \hat{a} \in C(\mathrm{Sp}(A))$, where $\hat{a}(\omega) := \omega(a)$. Here the topology on $\mathrm{Sp}(A) \subset A^*$ is the weak-star topology inherited from $A^*$. A net (= generalised sequence) $(\phi_\iota)_{\iota\in I}$ in $A^*$ converges w* to $\psi$ if it converges point-wise on $A$.
**Theorem 2** (Gelfand- Naimark) *For commutative C*-algebras $\Gamma$ is an isometric, bijective isomorphism between $A$ and $C(Sp(A))$, preserving multiplication and involution.*

**Proofs of the above results** (outline)
`ad (1)` Since $\|a^n\| \le \|a\|^n$, the series is absolutely convergent ($\Rightarrow$ convergent, by completeness of $A$). Preceding (1) five lines of remarks yield the result.
`ad (2)` If $x \in G_A$, then $y \in G_A \Leftrightarrow x^{-1}y \in G_A$ But $\|\mathbf{1} - x^{-1}y\| = \|x^{-1}(x-y)\| \le \|x^{-1}\|\|x-y\| < 1$ if $\|x-y\| < \frac{1}{\|x^{-1}\|}$. Hence the ball $\{y \in A : \ \|x-y\| < \frac{1}{\|x^{-1}\|}\}$ is contained in $G_A$.
`ad (3)` $\sigma(a)$ is closed as a pre-image of closed set $A \setminus G_A$ under isometry $\mathbb{C} \ni \lambda \to \lambda\mathbf{1} - a \in A$. If $|\lambda| > \|a\|$, then $x := \mathbf{1} - \frac{1}{\lambda}a$ satisfies $\|\mathbf{1} - x\| < 1$, hence $x$ and $\lambda x$ belong to $G_A$, so $\lambda \notin \sigma(a)$. Formula (*) follows directly from Carl Neumann's series.
`ad (4)` Continuity at $z_0 = \mathbf{1}$: by (1) we estimate that $\|z^{-1} - \mathbf{1}\| = \|\sum_{n=1}^{\infty}(\mathbf{1}-z)^n\| \le \sum_{n=1}^{\infty}\|(\mathbf{1}-z)\|^n = \frac{\|(\mathbf{1}-z)\|}{1-\|(\mathbf{1}-z)\|}$, converging to 0 as $z \to \mathbf{1}$.
Continuity at $x \in G_A$: if $x_n \to x$, then eventually $x_n \in G_A$ and $x_n x^{-1} \to xx^{-1} = \mathbf{1}$. By continuity at $\mathbf{1}$, $(x_n x^{-1})^{-1} = xx_n^{-1} \to 1$ and applying continuity of $L_{x^{-1}}$, we have $x^{-1}xx_n^{-1} \to x^{-1}\mathbf{1}$.
`ad (5)` By (3) if $|\lambda| > \|x\|$ then $\exists(\lambda\mathbf{1} - x)^{-1}$, hence $0 \neq \omega(\lambda\mathbf{1} - x) = \lambda - \omega(x)$, so $|\omega(x)| \le \|x\|$ and $\|\omega\| \le 1$. Since $|\omega(\mathbf{1})| = 1$, we have $\|\omega\| = 1$.
`ad (6)` Since $R_\lambda$ commutes with $R_\mu$, replacing $(\mu - \lambda)$ by $((\mu\mathbf{1} - a) - (\lambda\mathbf{1} - a))$ in the right -hand term of the equation, we verify it easily. Or you may subtract $R_\mu = (\lambda\mathbf{1} - a))R_\lambda R_\mu$ from equality $R_\lambda = (\mu\mathbf{1} - a)R_\lambda R_\mu$. Now (4) implies the continuity of $\lambda \mapsto f(\lambda)$. By linearity of $\psi$, Hilbert equation gives $\frac{f(\lambda)-f(\mu)}{\lambda-\mu} = -f(\lambda)f(\mu)$, which tends to $-f(\lambda)^2$, if $\mu \to \lambda$. Hence complex derivatives exist at all points of $\mathbb{C} \setminus \sigma(a)$.
`ad (7)` Assume towards a contradiction that $\sigma(a) = \emptyset$ for some $a \in A$. Then $\forall_{\psi\in A^*}\lambda \mapsto f(\lambda) := \psi(R_\lambda) \in \mathbb{C}$ are entire functions (clearly, non-zero). By (*) in (3), $|f(\lambda)| \le \|\psi\|\frac{1}{|\lambda|}\|(\mathbf{1} - \frac{1}{\lambda}a)^{-1}\|$, which goes to 0 as $\lambda \to \infty$, since the last term is bounded. Indeed, by (4), $(\mathbf{1} - \frac{1}{\lambda}a)^{-1}$ goes to $\mathbf{1}$ as $\lambda \to \infty$.
`ad (7a)` Suppose the contrary: that $\exists c := (ab)^{-1}$. Then $a(bc) = 1$ Also $1 = c(ba) = (cb)a$, so $a \in G_A$.
`ad (8)` By power series theory, $\sum_n c_n z^n$ converges absolutely in $\mathbb{D}_r$, so $\sum |c_n| r^n < \infty$. Consequently $\sum c_n a^n$ converges absolutely to $F(a) \in A$, since $\|a^n\| \le r^n$. Let us choose arbitrary $\lambda \in \sigma(a)$, so $|\lambda| \le r$. Algebraic formula for $x^n - y^n$ implies the following factorisation of $F(\lambda)\mathbf{1} - F(a) = \sum_{n=1}^{\infty} c_n(\lambda^n\mathbf{1} - a^n)$ as equal to $(\lambda\mathbf{1} - a)\sum_{n=1}^{\infty} c_n P_{n-1}(\lambda, a)$, where $P_{n-1}(\lambda, a) = \sum_{k=0}^{n-1} \lambda^k a^{n-k-1}$. Since $\|P_{n-1}(\lambda, a)\| \le nr^{n-1}$, the series $b := \sum_{n=1}^{\infty} c_n P_{n-1}(\lambda, a)$ converges in $A$ and its sum, $b$ commutes with $a$ (and with $(\lambda\mathbf{1} - a)$).That $F(\lambda)\mathbf{1} - F(a)$ is not invertible now follows from (7a).
`ad (8a)` We can write $p \in \mathbb{C}[z] \setminus \{0\}$ in the form $p(z) - \lambda = c(z - \lambda_1) \cdot \ldots \cdot (z - \lambda_n)$ for some constants $c, \lambda_j \in \mathbb{C}$, $c \neq 0$ (here $\{\lambda_1, \ldots \lambda_n\} = p^{-1}[\{\lambda\}]$ and in view of (7a), $p(a) - \lambda\mathbf{1} \notin G_A \Leftrightarrow \exists_j (a - \lambda_j\mathbf{1}) \notin G_A$. Reformulatin this as $\lambda \in \sigma(p(a)) \Leftrightarrow p^{-1}[\{\lambda\}] \cap \sigma(a) \neq \emptyset$, which in turn means $\lambda \in \sigma(p(a)) \Leftrightarrow \lambda \in p[\sigma(a)]$. If $p(z) = \frac{1}{z}$, then for $a \in G_A, \lambda \neq 0$ we hqave $(\frac{1}{\lambda}\mathbf{1} - a^{-1})(\lambda a) = a - \lambda\mathbf{1}$, so the non-invertibilities of $\frac{1}{\lambda}\mathbf{1} - a^{-1}$ and that of $a - \lambda\mathbf{1}$ are equivalent.
`ad (9)` Since $\sigma(a) \subset \{z \in \mathbb{C} : |z| \le \|a\|\}$, we have $r(a) \le \|a\|$. For $\lambda \in \sigma(a)$ we have $\lambda^n \in \sigma(a^n)$ by (8), so $|\lambda|^n \le \|a^n\|$, i.e. $|\lambda| \le \sqrt[n]{\|a^n\|}$ and $r(a) \le \inf_n \sqrt[n]{\|a^n\|}$.

Hence it suffices to show that $\limsup_n \sqrt[n]{\|a^n\|} \le r(a)$. Any functional $\varphi \in A^*$ applied to $R_\lambda$ represented by (*) in (3) gives a Laurant series $\varphi((\lambda\mathbf{1} - a)^{-1}) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}}\varphi(a^n)$ convergent for any $|\lambda| > r(a)$. Hence its terms are bounded for such $\lambda$, so there exists a constant $K_\varphi$ such that

[3]The inverse bijection: if $M$ is a max.ideal, then $\omega_M(a) = \lambda \Leftrightarrow (\lambda\mathbf{1} - a) \in M$ satisfies $\omega_M \in \mathrm{Sp}(A), M = \ker(\omega_M)$.

$\forall_n |\varphi(a^n)\lambda^{-n-1}| \leq K_\varphi$. By the (Banach-Steinhaus) *Uniform Boundedness Principle* $\forall_{|\lambda|>r(a)} \exists_C \forall_n \|a^n\||\lambda^{-n-1}| \leq C$. Here $C = C_\lambda$ does not depend on $n$, but may depend on $\lambda$ and since

$$\|a^n\|^{\frac{1}{n}} \leq C_\lambda^{\frac{1}{n}} |\lambda|^{\frac{n+1}{n}}, \quad \textit{we have} \quad \limsup_n \|a^n\|^{\frac{1}{n}} \leq |\lambda|$$

for any $|\lambda| > r(a)$, implying $\limsup_n \|a^n\|^{\frac{1}{n}} \leq r(a)$.
`ad (10)` Indeed, by the C*-condition $\|x^*xx^*x\| = \|x^*x\|^2 = \|x\|^4$ and since $xx^* = x^*x$, the left-hand term equals $\|(x^*)^2x^2\| = \|(x^2)^*(x^2)\| = \|x^2\|^2$. Hence $\|x^2\| = \|x\|^2$ and by induction, for any $k$ of the form $k = 2^m$, where $m \in \mathbb{N}$, $\|x^k\| = \|x\|^k$ Applying (9) to this subsequence we have $r(x) = \|x\|$.
`ad (11)` First we consider unitary elements $u \in A$, so that $u^* = u^{-1}$. It follows that $r(u) = \|u\| = \|u^{-1}\| = 1$ (Indeed, $\|u\|^2 = \|u^*u\| = \|\mathbf{1}\| = 1$.) Hence for any $\lambda \in \sigma(x)$ we have $|\lambda| \leq 1$ . Since $u^*$ is also unitary and $\omega(u^{-1}) = (\omega(u))^{-1}$ , we obtain even $|\omega(u)| = 1$ for any $\omega \in \mathrm{Sp}(A)$.

In the next step we use the fact that if $a, b \in A$ commute, then $\exp(a)\exp(b) = \exp(a+b)$ - here $\exp(a) := \sum_{n=0}^{\infty} \frac{1}{n!}a^n$. Its proof is a verbatim translation from the scalar case (when A=$\mathbb{C}$). If $a = a^*$ and $t \in \mathbb{R}$, we easily check that $(\exp(ita))^* = \exp(-ita)$ and consequently $\exp(ita)$ is unitary. Then for any $\omega \in \mathrm{Sp}(A), t \in \mathbb{R}$ we have $\omega(\exp(ita) \in \sigma(\exp(ita) \subset \{\lambda : |\lambda| \leq 1\}$. Hence $1 \geq |\omega(\exp(ita))| = |\exp(it\omega(a))|$, which is possible $\forall_{t\in\mathbb{R}}$ only if $\omega(a) \in \mathbb{R}$. Now for any $x \in A$ $x = a + ib$, where $a, b$ are self-adjoint (eg. $a = \frac{x+x^*}{2}$) and then $x^* = a - ib$, so using the fact that $\omega(a), \omega(b)$ are real, we obtain that $\omega(x^*) = \overline{\omega(x)}$.

`Proof of Theorem 1.` If $\lambda \in \sigma(x)$, (such $\lambda$ exists by (7)) then $\lambda\mathbf{1} - x \in A \setminus G_A = \{0\}$, so $x = \lambda\mathbf{1}$.

`Proof of Corollary 1.1` Note first that by ideal we mean here a proper ($\neq A$) subspace $I \subset A$ such that $a \in A \Rightarrow aI \subset I$. Clearly $\mathbf{1} \notin I$. Hence $G_A \cap I = \emptyset$ and the norm-closure of $I$ is also an ideal disjoint with $G_A$. Maximal ideals are therefore closed.

The quotient space $A/_I$ is also a unital algebra consisting of equivalence classes $[a] = a + I$ with $[a] \cdot [b] := [ab]$. The quotient map $\pi : A \ni a \mapsto [a] \in A/_I$ is a homomorphism of unital algebras. Here $\|[a]\| = \inf\{\|a - x\| : x \in I\}$ is the distance from $a$ to $I$ and is a sub-multiplicative norm, $\|[\mathbf{1}]\| = 1$. Easy verifications show that pre-image of an ideal $K \subset A/_I$ is an ideal in $A$ containing $I$ (and proper). When $I$ is maximal, we must have $\pi^{-1}(K) = I$, i.e. $K = \{[0]\}$. Hence $[0]$ is the only non-invertible element in $A/_I$ and by Theorem 1., we have

$$\forall_{a\in A} \exists!_{\lambda=\lambda_a\in\mathbb{C}} : [a] = \lambda[\mathbf{1}], \text{ i.e. } a - \lambda\mathbf{1} \in I.$$

The assignment $\omega_I : A \ni a \mapsto \lambda_a \in \mathbb{C}$ is linear and multiplicative, $\omega_I(\mathbf{1}) = 1$, hence $\omega_I \in \mathrm{Sp}(A)$. Also $I = \ker(\omega_I)$. Conversely, if $\omega \in \mathrm{Sp}(A)$, then $\ker(\omega)$ is an ideal $J$ having (as a linear subspace of $A$) co-dimension 1, which implies its maximality. Clearly, $\omega = \omega_J$ (as a pair of linear functionals having the same kernels and taking the same non-zero value at some vector $v$ (here $v = \mathbf{1}$).

In the proof of (5) we have seen that $\omega(a) \in \sigma(a)$. Conversely, if $\lambda \in \sigma(a)$, then $b := \lambda\mathbf{1} - a \notin G_A$. Then $bA := \{bx : x \in A\}$ is a proper ideal. By Kuratowski-Zorn's Lemma, such ideals are contained in some maximal ideal $I$. Then $\omega_I(b) = 0$, i.e. $\omega_I(a) = \lambda$.

`Proof of Theorem 2.` By property (11), the image $\Gamma(A)$ of $A$ under the Gelfand transform is a symmetric subalgebra of $C(\mathrm{Sp}(A))$, since $\omega(a^*) = \overline{\omega(a)}$, hence along with $\hat{a}$, this image contains the complex conjugate function $\widehat{a^*}$. Since by previous corollary

$$\sup\{|\hat{a}(\omega)| : \omega \in \mathrm{Sp}(A)\} = \sup\{|\lambda| : \lambda \in \sigma(a)\} = r(a),$$

the Gelfand transform $\Gamma$ is an isometry. Hence $\Gamma(A)$, as an isometric image of a complete space, is complete, conswquently -it is closed. It also contains all constant functions and separates the points of $\mathrm{Sp}(A)$. By Stone-Weierstrass Theorem, it equals $C(\mathrm{Sp}(A))$ and $\Gamma$ is surjective. Also $\Gamma$ is a homomorphism of unital C*-algebras. □

There is a version of Stone-Weierstrass Theorem for locally compact spaces $\Omega$ and symmetric subalgebras $A$ of $C_0(\Omega)$. $A$ is then non-unital, instead we assume that for any $\omega \in \Omega$ there exists $f \in A$ with $f(\omega) \neq 0$. Then theorem asserts that $A$ is dense in $C_0(\Omega)$.

Now a non-unital commutative C*-algebra $A$ is isometrically identified with $\{(a, 0) = a + 0 \cdot e : a \in A\} \subset A_1$ -a maximal ideal in $A_1$ and $\mathrm{Sp}(A_1)$ is the one-point compactification of $\mathrm{Sp}(A)$. One verifies that $\mathrm{Sp}(A) \cup \{0\}$ is a w*-closed subset of the (w*-compact) unit ball in $A_1^*$. Since $\{0\}$ is closed, $\mathrm{Sp}(A)$ is locally compact. For any $a \in A \subset A_1$ its Gelfand transform $\hat{a}$ is continuous on $\mathrm{Sp}A_1$ and vanishes on the ideal $A$. The corresponding to this ideal element $\varphi \in \mathrm{Sp}(A_1)$ is given by $\varphi(a + \lambda e) = \lambda$. The spectrum of $C_0(\Omega)$ is homeomorphic to $\Omega$ via evaluation functionals. The details are omitted, but simple.

# 1 Functional Calculus

Let $a \in A$ be a normal element in a unital C*- algebra. The (norm)-closure of polynomials in 3 variables: $a, a^*, \mathbf{1}$ is a commutative subalgebra of $A$, denoted[4] $C^*(a,1)$. For $x \in C^*(a,1)$ denote by $\sigma_*(x)$ its spectrum in this subalgebra. We shall prove that $\sigma_*(a) = \sigma(a)$ (the spectrum w.r. to $A$).

**Theorem 3.** *If $a \in A$ is normal then $C^*(a,1)$ is isometrically *-isomorphic to $C(\sigma(a))$. We also have unital isomorphism of C*-algebras $\Phi : C(\sigma(a) \ni f \mapsto f(a) \in C^*(a,1) \subset A$ that extends polynomial functional calculus in $a$, i.e. $\Phi(id) = a, \Phi(1) = \mathbf{1}$. Moreover*

$$\sigma(f(a)) = f(\sigma(a)),\ \|f\| = \|f(a)\| \quad \textit{and} \quad \widehat{f(a)} = f \circ \hat{a} \quad \textit{for } f \in C(\sigma(a)).$$

Here $id(\lambda) = \lambda, 1(\lambda) = 1$ and $\|f\| := \sup\{|f(\lambda)| : \lambda \in \sigma(a)\}$.

*Proof* First note that $\hat{a} : \mathrm{Sp}(C^*(a,1)) \ni \omega \mapsto \omega(a) \in \sigma_*(a)$ is surjective by Corollary 1.1, continuous and injective (hence a homeomorphism between these compact spaces). Its injectivity follows from the fact that if $\omega_1(a) = \omega_2(a)$ then since $\omega_j(a^*) = \overline{\omega_j(a)}, j = 1, 2$, these continuous, linear functionals agree on generators, whose products of powers together with 1 span a dense subset of $C^*(a,1)$. Now the composition operator $C(\sigma_*(a)) \ni f \mapsto f \circ \hat{a}$ composed with Gelfand transform's inverse $\Gamma^{-1} : C(\mathrm{Sp}(C^*(a,1))) \to C^*(a,1)$ establishes our isomorphism between $C(\sigma_*(a))$ and $C^*(a,1)$. Namely

$$\Phi(f) = \Gamma^{-1}(f \circ \hat{a}) \qquad (\text{hence } \Phi(id) = \Gamma^{-1}(\hat{a}) = a)).$$

Once we show that $\sigma_*(a) = \sigma(a)$, the remaining claims will be consequence of this isomorphism. The inclusion "$\supset$" is trivial, as $C^*(a,1) \subset A$, so inverse of $\lambda 1 - a$ in $C^*(a,1)$ will also be its inverse in $A$.

Suppose towards a contradiction that there exists $\alpha \in \sigma_*(a) \setminus \sigma(a)$. Then $b := (a - \alpha 1)^{-1}$ exists in $A$. The function $f : \sigma_*(a) \ni \lambda \mapsto \lambda - \alpha$ is continuous and for positive $\epsilon < \|b\|^{-1}$ the set $K := \{\lambda \in \sigma_*(a) : |f(\lambda)| \geq \epsilon\}$ is compact. Since $\alpha \notin K$, by Urysohn's Lemma, we find $g \in C(\sigma_*(a))$ vanishing on $K$, equal 1 at $\alpha$ and such that $0 \leq g \leq 1$. Clearly $\|fg\|$ - the sup-norm over $\sigma_*(a)$ is $\leq \epsilon$. By the above isomorphism, we have $g = \hat{\gamma}$ for sone $\gamma \in C^*(a,1)$ and $b(a - \alpha 1)\gamma = \gamma$ has norm 1, since the spectrum of $\gamma$ is the range of $g$. But this contradicts the estimate $\|b(a-\alpha 1)\gamma\| \leq \|\hat{b}\|\|(\hat{a}-\alpha)g\| < 1$.

The functional calculus in $a$ is the inverse mapping to the isomorphism: $C^*(a,1) \to C(\sigma(a))$. $\square$.

More general case has a quite similar proof: if $A$ is a commutative C*-algebra containing $a_1, \ldots, a_n$, one can define their **joint spectrum** in $A$ as the set $\sigma_A(a_1, \ldots, a_n) \subset \mathbb{C}^n$ consisting of those $n$-tuples $(\lambda_1, \ldots, \lambda_n)$, for which the ideal generated in $A$ by $\lambda_1 \mathbf{1} - a_1, \ldots, \lambda_n \mathbf{1} - a_n$ is proper. Now *if these elements generate $A$ as a C*-algebra, then their "joint Gelfand transform": $Sp(A) \ni \omega \mapsto (\omega(a_1), \ldots, \omega(a_n))$ maps the spectrum of $A$ homeomorphically onto $\sigma_A(a_1, \ldots, a_n)$ and $A$ is isomorphic with the C*- algebra of continuous functions on this joint spectrum of $(a_1, \ldots, a_n)$.*

In particular, unital subalgebras generated by a self-adjoint element $a$ in C* algebra can be identified with $C(\sigma(a))$, where $\sigma(a) \subset \mathbb{R}$ by Corollary (11) and $\sigma(x^*x)$, as the range of $\overline{\hat{x}}\hat{x}$ is a subset of $[0, +\infty) = \mathbb{R}_+$ if $x$ is normal. We say that $a \in A$ is a **positive element**, if $a = a^*$ and $\sigma(a) \subset \mathbb{R}_+$. Note that 0 is also positive. Functional calculus using functions $t \mapsto t_+ := \max(t, 0)$ (resp.$t_- = (-t)_+, t \in \mathbb{R}$ allows us to write any $x = x^* \in A$ as a difference $x = x_+ - x_-$ of two positive elements. By applying the above isomorphic description of $C^*(a,1)$ we have the following useful characterisation:

**Lemma 4**. *For a self-adjoint element $a \in A \setminus \{0\}$ the following are equivalent:*

(i) $a$ is positive,

(ii) $\|\frac{1}{\|a\|}a - \mathbf{1}\| \leq 1$, i.e. $\|\|a\|\mathbf{1} - a\| \leq \|a\|$,

(iii) $a = x^*x$ for some $x \in A$ ( namely, for $x = \sqrt{a}$ )[5]

The second condition can be applied to show that **sum of two positive elements is positive**, [6] a fact used in proof that $(iii) \Rightarrow (i)$ (here the commutativity of $x, x^*$ is not assumed!). Also note that

$$(\dagger) \qquad \forall_x\, (x^* = x \Rightarrow \|x\|\mathbf{1} - x \quad \text{is positive}).$$

If $a$ is positive, ten so is $x^*ax$ (for all $x \in A$), since $x^*ax = (a^{\frac{1}{2}}x)^*(a^{\frac{1}{2}}x)$.

**Definition** A linear functional $\varphi : A \to \mathbb{C}$ on a C*-algebra $A$ is **positive**, in symbols, $\varphi \geq 0$, if its values on positive elements are non-negative: $\forall_{x \in A}\, \varphi(x^*x) \geq 0$. **States** of $A$ are positive functionals having value 1 at $\mathbf{1}$. **Pure states** are extreme points in the set of all states on $A$.

By $(\dagger)$ we have $\varphi(x^*x) \leq \|x^*x\|\varphi(1)$, if $\varphi \geq 0$.

**Properties of positive functionals** For any $x, y \in A$ and $\varphi : A \to \mathbb{C}$-positive we have

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[4] If $E \subset A$, one denotes by $C^*(E)$ the smallest C*-subalgebra of $A$ containing $E$ (i.e. generated by $E$).

[5] The proof that (iii) implies (i) is a bit tricky, but for algebras of operators it follows easily from the fact that the closure of numerical range contains the spectrum.GNS construction described later on can be applied in general case.

[6] If $a, b \in A^+$, then for $\alpha := \|a\|, \beta := \|b\|, \gamma := \alpha + \beta, c = a + b$, we have by (ii) that $\|\gamma\mathbf{1} - c\| \leq \|\alpha\mathbf{1} - a\| + \|\beta\mathbf{1} - b\| \leq \gamma$. Since $c = c^*$, this gives $\sigma(\gamma\mathbf{1} - c) \subset [-\gamma, \gamma]$, which implies $\sigma(c) \subset [0, 2\gamma]$.

1. $\varphi(x^*) = \overline{\varphi(x)}$ (we say that functionals satisfying 1. are hermitian)
2. $|\varphi(y^*x)|^2 \leq \varphi(x^*x)\varphi(y^*y)$,
3. $|\varphi(x)|^2 \leq \varphi(1)\varphi(x^*x)$
4. $\|\varphi\| = \varphi(1)$.
5. *Conversely, if* $f \in A^*$ *and* $f(a) = \|f\|\|a\|$ *for some* $a = x^*x \in A \setminus \{0\}$, *then* $f \geq 0$.

To prove 5., without loss of generality assume that $\|f\| = 1 = \|a\|, f(a) = 1$. To show first that $f(1) \in \mathbb{R}$ assume that $f(1) = \alpha + i\beta, \alpha, \beta \in \mathbb{R}$. If $\beta \neq 0$, then $\forall_{t\in\mathbb{R}} |f(1+tia)| = |\alpha + i(t+\beta)| \geq |t+\beta|$ But $|f(1+tia)| \leq \|1+tia\| \leq \sqrt{1+t^2}$. Hence $|t+\beta| \leq \sqrt{1+t^2}$, i.e. $t^2 + 2t\beta + \beta^2 \leq 1 + t^2$ for all $t \in \mathbb{R}$ - a contradiction. Hence $f(1)$ is real. Suppose $f(1) < 1$. Then $|f(1-2a)| = |f(1) - 2| > 1$ and $|f(1-2a)| \leq \|1-2a\| = \|(1-a)-a\| \leq 1$ -a contradiction. Hence $f(1) = 1$.

Now if $b = b^*$ suppose that $f(b) = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$. If $\beta \neq 0$, then $\forall_{t\in\mathbb{R}} |f(b+it1)| \geq |\beta + t|$ and $|f(b+it1)| \leq \|b+it1\| = \sqrt{\|b\|^2 + t^2}$, a contradiction. Hence $f(b) \in \mathbb{R}$ if $b = b^*$. Now take any $h = h^* \geq 0$ and suppose that $f(h) < 0$. Then $f(1-h) = 1 - f(h) > 1$, but this contradicts $|f(1-h)| \leq \|1-h\| \leq 1$. □.

In particular, if $K$ is a compact Hausdorff space, then the set of states of $C(K)$ is the set of probabilistic regular Borel measures on $K$ and pure states are delta-measures.

If $\omega : A \to \mathbb{C}$ is a state, then $A \times A \ni (a,b) \mapsto \langle a|b\rangle := \omega(b^*a)$ is a sesquilinear form, obeying the **Schwarz inequality:** $|\langle a|b\rangle|^2 \leq \langle a|a\rangle\langle b|b\rangle$ for $a, b \in A$. Hence $\mathcal{N}_\omega := \{x \in A : \omega(x^*x) = 0\}$ is a closed left ideal. The quotient space $H^\circ_\omega$ is a pre-Hilbert space with inner product $\langle [a]|[b]\rangle_\omega := \omega(b^*a)$ and it is a dense subspace of its completion denoted by $H_\omega$. The "left multiplication" operators $\pi_\omega(a) : H^\circ_\omega \ni [b] \mapsto [ab]$ extend by continuity on $H_\omega$ and $A \ni a \mapsto \pi_\omega(a) \in \mathcal{B}(H_\omega)$ is a representation (i.e. a unital *-homomorphism of C*-algebras). One shows that the direct (orthogonal) sum

$$\bigoplus_\omega \pi_\omega : A \to \mathcal{B}(H_\omega),$$

where $\omega$ runs through the set of all states on $H$ –is an isometric *-homomorphism called the **universal representation of** $A$. This (so called) **GNS construction** allows us to treat any C*-algebra as an algebra of operators on some Hilbert space.

Here we concentrate on commutative (subalgebras of) C*-algebras in order to obtain spectral theorem for normal operators. We begin by an important fact on **commutants**[7] $\mathcal{R}'$ of sets $\mathcal{R} \subset \mathcal{B}(H)$. **Definition** $\mathcal{R}' := \{S \in \mathcal{B}(H) : \forall_{T\in\mathcal{R}} TS = ST\}$. We say that $S, T$ doubly commute if $ST = TS$ and $S^*T = TS^*$. (Clearly $\mathcal{R}'$ is a unital subalgebra of $\mathcal{B}(H)$, closed in WOT-topology :=the topology defined on $\mathcal{B}(H)$ by the family of semi-norms $p_{x,y}(T) := |\langle Tx|y\rangle|, x\, y \in H$.)

It is not obvious, but for a normal operator $T \in \mathcal{B}(H)$ its commutant, $\{T\}'$ is conjugate -closed: Operators commuting with $T$ also doubly commute with $T$, as follows from the following more general **Theorem 5. (Fuglede-Putnam)** *If* $M, N \in \mathbf{B}(H)$ *are normal and "intertwined by* $T \in \mathcal{B}(H)$*" i.e. such that* $MT = TN$, *then* $M^*T = TN^*$.

Proof. Note that T also intertwines $M^k$ with $N^k$ for $k \in \mathbb{N}$ and $T$ intertwines $\exp(M) := \sum_{k=0}^\infty \frac{1}{k!} M^k$ with $\exp(N)$. Hence $T = \exp(-M)T\exp(N)$. Multiplying this equality by $U := \exp(M - M^*) = \exp(M)(\exp(M^*))^{-1}$ from the left and by $W := (\exp N)^{-1}\exp(N^*)$ from the right, we have $UTW = \exp(M^*)T\exp(-N^*)$. Since $V^* = -V$ for $V := M - M^*$, we have $U^* = (\exp(V))^* = \exp(-V) = \exp(V)^{-1}$, operators $U$ and $W$ are unitary, hence having norm 1. Consequently, $\|\exp(M^*)T\exp(-N^*)\| = \|UTW\| \leq \|T\|$. Replacing $N$ and $M$ by $zN$ , $zM$, where $z \in \mathbb{C}$ is arbitrary, we deduce the boundedness of (analytic) function $z \mapsto F(z) := \exp(zM^*)T\exp(-zN^*)$. As in the proof of (7), by Liouville's theorem, $F$ must be constant equal $F(0) = T$. Hence $\exp(zM^*)T = T\exp(zN^*)$. Comparing the coefficients at $z^1$ we obtain the result. □

**Corollary 5.1** Similar normal operators are unitarily equivalent. This means that if $N, M \in \mathcal{B}(H)$ are normal and for some bijective $S \in B(H)$ we have $SM = NS$, then there exist a unitary $U$ with $UM = NU$.

**Proof**: From the above theorem, $SM^* = N^*S$ and by taking adjoints, $MS^* = S^*N$. Combined with our assumption this yields $S^*SM = S^*NS = MS^*S$, so $M$ commutes with $|S| = (S^*S)^{\frac{1}{2}}$. Since $S$ is bijective, $U$ in its polar decomposition $S = U|S|$ –is unitary. Now $UMU^* = U|S|M|S|^{-1}U^{-1} = SMS^{-1} = N$. □

[7] called also centralizers, denoted in W.Rudin's book by $\Gamma(\mathcal{R})$ instead of $\mathcal{R}'$. The reason is that "prime" is often used there to denote the dual space $X'$ for a Banach space $X$, but if only operators on Hilbert spaces are considered, most authors use "prime " for commutators. In these notes $\Gamma$ denotes a different thing: the Gelfand transform.

## 2 Spectral Theorems

**Definition.** A sequence of self-adjoint operators $T_n \in \mathcal{B}(H)$ is *monotone increasing* if $T_n \leq T_{n+1}$ for all $n$ and *bounded* if for some $C \in \mathbb{R}_+$ we have $T_n \leq C\mathbf{1}$, where $\mathbf{1}(x) = x, x \in H$.

We have the following result:

**Lemma 6.** *For all bounded monotone sequences $(T_n)$ of self-adjoint operators there exists $T \in \mathcal{B}(H)$ such that $\|Tx - T_n x\| \to 0$ as $n \to \infty$. (Analogous result holds for monotone nets).*

Indeed, as bounded and monotone, the sequence $\langle T_n x|x\rangle$ converges in $\mathbb{R}$ for any $x \in H$. By polarisation, so do the sequences $\langle T_n x|y\rangle \in \mathbb{C}$ if $x, y \in H$, their limits define a bounded sesquilinear form corresponding to a bounded operator $T = T^*$. Hence $T_n \to T$ in WOT topology. The SOT convergence follows since

$$\|(T - T_n)x\|^2 = \langle (T - T_n)^2 x|x\rangle \leq \|T - T_n\|\langle (T - T_n)x|x\rangle \to 0 \quad (\text{as } n \to \infty).$$

Here we are using the obvious inequality $S^2 \leq \|S\|S$ for positive elements $S$ in commutative C*-algebras identified with $C(\Omega)$.

**Definition.** By **spectral measure**[8] in a Hilbert space $H$ defined on a sigma field $\mathfrak{M}$ of subsets of $\Omega$ we understand the mapping $E : \mathfrak{M} \to \mathcal{B}(H)$ having the following properties: $E(\emptyset) = 0$, $E(\Omega) = \mathbf{1}$, $\forall_{\Delta\in\mathfrak{M}} E(\Delta)^* = E(\Delta)^2 = E(\Delta)$ and $\mu_x(\Delta) := \langle E(\Delta)x|x\rangle$ is $\sigma$-additive for all $x \in H$.

Note that by polarisation formula, also $\mu_{x,y} : \Delta \mapsto \langle E(\Delta)x|x\rangle$ are countably additive (i.e. $\mu_{x,y}$ are complex measures). From Lemma 6. we obtain countable additivity of $\Delta \mapsto E(\Delta)$ not only in WOT, but in strong operator topology as well.

Let $L^\infty(\mathfrak{M})$ be the Banach algebra of bounded, $\mathfrak{M}$ –measurable functions $f : \Omega \to \mathbb{C}$ (with sup-norm). For $f \in L^\infty(\mathfrak{M})$ the (bounded, sesquilinear) form $H \times H \ni (x, y) \mapsto \int_\Omega f d\mu_{x,y}$ defines a bounded linear operator denoted $\int f(\omega)\, E(d\omega)$, or $\int_\Omega f\, dE$ and called the **spectral integral of $f$ w.r. to** $E$. There is an alternative way of defining such integrals:

(a1) For characteristic functions $\int \chi_\Delta\, dE := E(\Delta)$.

(a2) This together with linearity requirement forces one to define $\int \sum_{k=1}^n c_k \chi_{\Delta_k}\, dE := \sum_{k=1}^n c_k E(\Delta_k)$.

(a3) Then using uniform approximation of $f$ by simple functions $f_n$ one defines $\int f\, dE$ for $f \in L^\infty(\mathfrak{M})$ as the norm-limit of the corresponding integrals of $f_n$.

Verification that two approaches are equivalent is simple (using Lebesgue's Dominated Convergence Theorem for $\int f_n d\mu_{x,y}$.) The correspondence $f \mapsto \int_\Omega f\, dE \in \mathcal{B}(H)$ is **not only linear, but multiplicative** as well. This key fact relies on the following property of orthogonal projections:

**Lemma.** If $P, Q$ are orthogonal projections in $H$, then the following are equivalent:

(i) $(P + Q)^2 = P + Q$ (i.e. $P + Q$ is a projection),

(ii) $PQ = 0$,

(iii) $M := P(H) \perp Q(H) =: N$.

*Proof.* Since $\|Pf\|^2 + \|Qf\|^2 = \langle Pf|f\rangle + \langle Qf|f\rangle = \langle (P + Q)f|f\rangle \leq \|f\|^2$ (by (i)), applying this for $f = Ph$, we have $\|PPh\|^2 + \|QPh\|^2 \leq \|Ph\|^2$ and since $PPh = Ph$, (i) implies that $QPh = 0$. Next, (ii) implies that for $f \in N$ we have $0 = PQf = Pf$, implying that $f = Qf \perp M$, since $\ker P = M^\perp$. Implication (iii)⇒(i) is verified by an easy computation. □

Now if $\Delta_1, \Delta_2 \in \mathfrak{M}$ are disjoint, then $E(\Delta_1) + E(\Delta_2)$ is a projection, hence $E(\Delta_1)E(\Delta_2) = 0$. An easy computation for arbitrary $\Delta_1, \Delta_2 \in \mathfrak{M}$ with $\Delta := \Delta_1 \cap \Delta_2$ after cancelling products of values of $E$ at disjoint pairs of sets (namely $(\Delta_1 \setminus \Delta, \Delta_2 \setminus \Delta)$; $(\Delta_1 \setminus \Delta, \Delta)$ and $(\Delta, \Delta_2 \setminus \Delta)$) yields

$$(\ddagger) \qquad E(\Delta_1)E(\Delta_2) = \Big(E(\Delta_1 \setminus \Delta) + E(\Delta)\Big)\Big(E(\Delta_2 \setminus \Delta) + E(\Delta)\Big) = E(\Delta)^2 = E(\Delta).$$

Therefore (because $\chi_{\Delta_1} \cdot \chi_{\Delta_2} = \chi_{\Delta_1\cap\Delta_2}$), we have multiplicativity at stage (a1) of our construction of the spectral integral. By taking intersections, we may represent a pair $(g, h)$ of simple functions as a pair of linear combinations of characteristic functions from a common partition $(\Delta_k)_{k\leq n}$ of $\Omega$ into pairwise disjoint measurable sets: $g = \sum_{k=1}^n a_k \chi_{\Delta_k}, h = \sum_{k=1}^n b_k \chi_{\Delta_k}$ and then (taking into account only non-empty intersections) we have $fg = \sum_{k=1}^n a_k b_k \chi_{\Delta_k}$. Computing product $(\int g\, dE)(\int h\, dE)$ and again taking into account only non-empty intersections, due to (‡), we see that it equals $\int gh\, dE$. Passing to stage (a3) we only use continuity of the product with respect to operator norm.

Since $\int f\, dE$ depends linearly on the function $f$ and $\int \bar{f}\, dE = (\int f\, dE)^*$ (by similar reasons - starting with simple functions), we have the following

[8] Spectral measures are also called "resolutions of identity". Note that $\langle E(\Delta)x|x\rangle = \|E(\Delta)x\|^2$

**Proposition 7.** *The mapping $\Phi : L^\infty(\mathfrak{M}) \ni f \mapsto \int f\,dE \in \mathcal{B}(H)$ is a *-homomorphism between C*-algebras. In particular, $\|\int f\,dE\| \le \|f\|_\Omega := \sup\{|f(\omega)| : \omega \in \Omega\}$.*

However to obtain an isomorphism we have to consider "equality almost everywhere $[dE]$". There exists a countable family $\{D_n : n \in \mathbb{N}\}$ of open disks forming a base of topology in $\mathbb{C}$. For a measurable (w.r. to $\mathfrak{M}$) function $f : \Omega \to \mathbb{C}$ we define its **essential range** to be the intersection:

$$E - \text{ess rg}(f) := \bigcap\{\mathbb{C} \setminus D_n : E(f^{-1}(D_n)) = 0\}.$$

We say that a property $P(\omega)$ holds $E$-a.e. (almost everywhere), if the set $N_P := \{\omega \in \Omega : P(\omega) \text{ does not hold }\}$ is contained in some $\Delta \in \mathfrak{M}$ such that $E(\Delta) = 0$. Then $E-$ess rg$(f)$ is the smallest closed set in $\mathbb{C}$ that contains $f(\omega)$ for almost every $\omega \in \Omega$. Let $\|f\|_\infty := \sup\{|\lambda| : \lambda \in E-\text{ess rg}(f)\}$. This is a semi-norm (also sub-multiplicative) and $N_E := \{f \in L^\infty(\mathfrak{M}) : \|f\|_\infty = 0\} = \{f : f = 0\ \text{E}-a.e.\}$ is a closed ideal in $L^\infty(\mathfrak{M})$. we define $L^\infty(E)$ as the quotient algebra $L^\infty(\mathfrak{M})/_{N_E}$ normed by $\|[f]\|_\infty := \|f\|_\infty$, where $[f] = f + N_E$ is the equivalence class of $f$. It is easy to show that neither $\|f\|_\infty$, nor $\int f\,dE$ depend on the choice of representative $f$ of the equivalence class $[f]$, allowing us to use simplified notation $f$ instead of $[f]$ for elements of $L^\infty(E)$. We have the following refinement of Proposition 7 (which we leave now without proof):

**Theorem 8.** *Let $E : \mathfrak{M} \to \mathcal{B}(H)$ be a spectral measure. The mapping $L^\infty(E) \ni f \to \Psi(f) := \int f\,dE$ is an isometric *-isomorphism from $L^\infty(E)$ onto a commutative, unital C* -subalgebra of $\mathcal{B}(H)$. Moreover $\|\Psi(f)x\|^2 = \int |f|^2\,d\mu_x$ and $\sigma(\Psi(f)) = E - ess\ rg(f)$.* □

From now on, $\mathfrak{M}$ will be the sigma-field of Borel subsets of a compact Hausdorff space $\Omega$. In the case when $\Omega = \sigma(T)$, we say that $E$ is a spectral measure for $T \in \mathcal{B}(H)$ if for $f_1(\lambda) = \lambda$ we have $T = \int_{\sigma(T)} \lambda\, E(d\lambda)$.

Our main goal is the construction of spectral measure for a normal operator $T$. We shall consider even more general construction representing a commutative, unital C*-subalgebra $A$ of $B(H)$ applicable for representing commuting families of normal operators.

One of the key techniques of the proof will be the following comparison principle for two measures based on the by uniqueness part of Riesz-Markof-Kakutani Theorem (RMK-for short):

**Lemma C.** *Given $\rho \in L^\infty(\mathfrak{M})$ and a Borel complex measure $\mu$ on $\Omega$ define $\rho \cdot \mu$ by equality $\rho \cdot \mu(\Delta) = \int_\Delta \rho\, d\mu$. Then for a Borel complex measure $\mu$ on $\Omega$ we have*

$$\nu = \rho \cdot \mu \Leftrightarrow \Big(\forall_{f \in C(\Omega)} \int f\,d\nu = \int f\rho\,d\mu\Big).$$

*If $\nu = \rho \cdot \mu$, the last equalities hold also for all $f \in L^\infty(\mathfrak{M})$.*

**Theorem 9.** *Let $\Omega = Sp(A)$ be the spectrum of a commutative, unital C*-subalgebra $A$ of $B(H)$.*
*(i) Then there exists a unique Borel spectral measure $E$ on $\Omega$ such that*

$$T = \int_\Omega \hat{T}\,dE \quad \text{for any } T \in A, \tag{e9}$$

*where $\hat{T} = \Gamma(T) \in C(\Omega)$ is the Gelfand transform of $T$.*
*(ii) The C*-algebra homomorphism $\Phi : L^\infty(\mathfrak{M}) \ni f \mapsto \int f\,dE$ acting onto a closed C*-subalgebra $B$ of $\mathcal{B}(H)$ extends the mapping $\Gamma^{-1} : C(\Omega) \to A$.*
*(iii) If $U \neq \emptyset, U \subset \Omega$ is w*- open, then $E(U) \neq 0$.*
*(iv) An operator $S \in \mathcal{B}(H)$ commutes with $A$ iff[9] it commutes with all $E(\Delta)$, $\Delta \in \mathfrak{M}$.*

Proof. ad (i): By Gelfand-Naimark Theorem, $\Gamma^{-1} : C(\Omega) \ni \hat{T} \mapsto T \in A$ is a *-isomorphism. Hence (using (RMK) Theorem) given $x, y \in H$ we find complex Borel measures $\mu_{xy}$ on $\Omega$ such that $\int \hat{T}\,d\mu_{x,y} = \langle Tx|y\rangle$ for any $T \in A$. The correspondence: $(x, y) \mapsto \mu_{x,y}$ is linear in $x$ and anti-linear in $y$, i.e. for any Borel set $\Delta \subset \Omega$ the mapping: $H \times H \ni (x, y) \mapsto \mu_{x,y}(\Delta)$ is a sesqui-linear form. This form is bounded, since $|\langle Tx|y\rangle| \le \|T\|\|x\|\|y\|$ and $\|T\| = r(T) = \|\hat{T}\|_\Omega$. The norm of functional $C(\Omega) \ni f \mapsto \int f\,d\mu_{x,y}$ is the total variation norm of $\mu_{x,y}$. By the 1-1 correspondence between bounded sesqui-linear forms and operators, there is a unique operator $\Phi(f)$ such that $\langle \Phi(f)x|y\rangle = \int f\,d\mu_{x,y}$ for all $x, y \in H$. In particular, for any $\Delta \in \mathfrak{M}$ there exists a unique operator $E(\Delta) \in \mathcal{B}(H)$ such that

$$\langle E(\Delta)x|y\rangle = \mu_{x,y}(\Delta) \quad \text{for any } x, y \in H.$$

Since $\Gamma$ and its inverse are *-morphisms, we have $\Gamma^{-1}(\bar{f}) = (\Gamma^{-1}(f))^*$ for any $f \in C(\Omega)$ and consequently $T \in A$ is self-adjoint iff $\hat{T}(\Omega) \subset \mathbb{R}$ and for such $T$ we have $\int \hat{T}\,d\mu_{x,y} = \langle Tx|y\rangle = \langle x|Ty\rangle = \overline{\langle Ty|x\rangle} = \overline{\int \hat{T}\,d\mu_{y,x}}$. Hence $\mu_{y,x} = \overline{\mu_{x,y}}$ (by uniqueness claim of (RMK) Theorem). Hence

[9] "iff" is an abbreviation of "if and only if"

$E(\Delta)$ are self-adjoint. Since $\mu_{x,y}$ are sigma-additive, so is $\Delta \mapsto E(\Delta)$ (in WOT-topology). Once we show that they are projections, the remaining claims concerning $\Phi$ will be consequences of Lemma 6. and Proposition 7. (To show that $B = \Phi(L^\infty(\mathfrak{M}))$ is closed one needs to verify that $B$ is also the range of $\Psi$ and to apply isometry of $\Psi$ asserted in Theorem 8 and completeness of $L^\infty(E)$. For this part of proof the reader may consult W. Rudin's Functional Analysis book)

To show (iii) suppose, that on the contrary $E(U) = 0$ and take a non-zero $f \in C(\Omega)$ with support contained in $U$. Then $0 \neq T_0 := \Gamma^{-1}(f)$, but we have $T_0 = \int_\Omega f\, dE = \int_U f\, dE = 0$, a contradiction.

ad (iv): For $S, T \in A$, we have

$$\int_\Delta \hat{S}\hat{T} d\mu_{x,y} = \langle STx|y\rangle = \int_\Omega \hat{S} d\mu_{Tx,y}.$$

Applying Lemma C and the fact that any $f \in C(\Omega)$ is of the form $\hat{S}$ for some $S \in A$, we obtain for any $f \in L^\infty(\mathfrak{M})$ that

$$\int f\hat{T} d\mu_{x,y} = \langle \Phi(f)Tx|y\rangle = \langle Tx|(\Phi(f))^* y\rangle = \int \hat{T} d\mu_{x,(\Phi(f))^* y}.$$

By analogous reason we may replace here $\hat{T}$ by an arbitrary $g \in L^\infty(\mathfrak{M})$ obtaining for such $g$ that

$$\langle \Phi(fg)x|y\rangle = \int fg\, d\mu_{x,y} = \int g\, d\mu_{x,(\Phi(f))^* y} = \langle \Phi(g)x|(\Phi(f))^* y\rangle = \langle \Phi(f)\Phi(g)x|y\rangle,$$

which implies that $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in L^\infty(\mathfrak{M})$. Applying this for $f = g = \chi_\Delta$, we have shown that $E(\Delta)$ (which is $\Phi(\chi_\Delta)$) equals its square, hence it is a projection.

Finally, if $S \in \mathcal{B}(H)$ commutes with any $T \in A$, we have for any $x, y \in H$ and for $z := S^* y$ that $\langle STx|y\rangle = \langle Tx|z\rangle = \int \hat{T} d\mu_{x,z} = \langle TSx|y\rangle = \int \hat{T} d\mu_{Sx,y}$ and again by uniqueness claim of (RMK) theorem, $\mu_{x,z} = \mu_{Sx,y}$. Then for any $\Delta \in \mathfrak{M}$, $\langle SE(\Delta)x|y\rangle = \langle E(\Delta)x|z\rangle = \mu_{x,z}(\Delta)$ and $\langle E(\Delta)Sx|y\rangle = \mu_{Sx,y}(\Delta)$, hence $SE(\Delta) = E(\Delta)S \Leftrightarrow \mu_{x,z} = \mu_{Sx,y}$ . By reversing this argument we obtain the opposite implication: $SE(\Delta) = E(\Delta)S \Rightarrow ST = TS$ $(\forall T \in A)$. □

As a corollary, we obtain spectral theorem for normal operators.

**Theorem 10.** *For a normal operator $T \in \mathcal{B}(H)$ there exists a unique Borel spectral measure on $\sigma(T)$ such that*

$$T = \int_{\sigma(T)} \lambda\, E(d\lambda). \tag{e10}$$

*An operator $S \in \mathcal{B}(H)$ commutes with $T \Leftrightarrow SE(\Delta) = E(\Delta)S$ for any Borel set $\Delta \subset \sigma(T)$.*

Proof. We apply Theorem 9. for $A$, the (commutative) C*-algebra generated by $T, I$, whose maximal ideal space is identified with $\sigma(T)$ by the homeomorphism $\hat{T}$ ( the Gelfand transform of $T$, as in the proof of Theorem 3. Any point $\lambda \in \sigma(T)$ is the image under the bijection $\hat{T}$ of a unique element $\omega_\lambda$ of $\Omega := \mathrm{Sp}(A)$. For any Borel set $\Delta \subset \sigma(T)$ the set $\hat{T}^{-1}(\Delta) = \{\omega_\lambda : \lambda \in \Delta\}$ is a Borel subset of $\Omega$ and if $E_A$ is the spectral measure for $A$ constructed in Theorem 9., then $E(\Delta) := E_A(\hat{T}^{-1}(\Delta))$ is the spectral measure on $\sigma(T)$. Formally, $E$ is the transport (or the image) of measure $E_A$ under the mapping $\hat{T}$ and using for integrals of simple functions $\chi_\Delta$ the obvious equality $\chi_{\hat{T}^{-1}(\Delta)} = \chi_\Delta \circ \hat{T}$ and then passing to continuous functions (which are uniform limits of simple functions) we see that

$$\int_{\sigma(T)} f(\lambda)\, E(d\lambda) = \int_\Omega (f \circ \hat{T})(\omega)\, E_A(d\omega).$$

Since $f_1(\lambda) = \lambda$ acts as identity on the range of $\hat{T}$, we have $f \circ \hat{T} = \hat{T}$, the equation (e9) of the last theorem implies that $T = \int \lambda\, E(d\lambda)$.

To show the uniqueness, note first that by *-homomorphism of the spectral integrals (Proposition 7) or by Theorem 8, for $f(z) = \sum_{k=0}^n (a_k \bar{z}^k + b_k z^k)$ - a complex polynomial in $(z, \bar{z})$, we have $\int f(z)\, E(dz) = \sum_{k=0}^n (a_k (T^*)^k + b_k T^k)$ for any spectral measure $E$ satisfying (e10). By Stone-Weierstrass Theorem, such functions are uniformly dense in $C(\sigma(T))$. This implies[10] the uniqueness of such $E$.

If $ST = TS$, then by Theorem 5., also $ST^* = T^* S$ and consequently, $S$ commutes with any element of $A = C^*(T, I)$. By the last part of Theorem 9, $S$ commutes with each $E(\Delta)$. Conversely, if $S$ commutes with $E(\Delta)$, it commutes with spectral integrals of simple functions and the latter functions approximate $f_1(\lambda) = \lambda$ uniformly on $\sigma(T)$. Hence $ST = TS$. □

[10]This uniqueness is verified first for the measures $\mu_x(\Delta) := \langle E(\Delta)x|x\rangle$

If $f \in L^\infty(\mathfrak{M})$, then instead of $\Phi(f)$ we will also write $f(T)$, calling $\Phi$ the extended functional calculus in a normal operator $T$. Since for $f_1 \in C(\sigma(T))$, where $f_1(\lambda) = \lambda$ we have $T^* = \bar{f}_1(T)$, $T^*T = (|f_1|^2)(T)$ and $\chi_\Delta(T) = E(\Delta)$ we have the following

**Corollary 11.** *If $T$ is a normal operator than $T$ is self-adjoint iff $\sigma(T) \subset \mathbb{R}$ and $T$ is unitary iff $\sigma(T)$ is a subset of the unit circle $\partial D = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. If for $f \in C(\sigma(T))$ we define $\Delta_0 := f^{-1}(\{0\})$, then for $x \in H$ we have $\|f(T)x\|^2 = \langle (f(T))^* f(T)x|x\rangle = \int |f|^2 \, d\mu_x$ and*

$$f(T)x = 0 \Leftrightarrow x \in \mathcal{R}(E(\Delta_0)) := E(\Delta_0)(H).$$

PROOF. Only the last equivalence needs now to be explained. If $g$ is the characteristic function of $\Delta_0$, then $fg = 0$, so $0 = f(T)g(T) = f(T)E(\Delta_0)$, showing that "$\Leftarrow$" is true. If $\Delta_n := \{\lambda \in \sigma(T) : |f(\lambda)| \geq \frac{1}{n}\}$, then $\Delta_* := \bigcup_{n\in\mathbb{N}} \Delta_n = \sigma(T) \setminus \Delta_0$ and $f_n(\lambda)$ equal $\frac{1}{f(\lambda)}$ for $\lambda \in \Delta_n$ and equal 0 outside $\Delta_n$ is in $L^\infty(\mathfrak{M})$ and such that $f_n f = \chi_{\Delta_n}$, hence $f_n(T)f(T) = E(\Delta_n)$. Now if $f(T)x = 0$, then $E(\Delta_n)x = 0$ and by sigma-additivity, also $E(\Delta_*)x = 0$. Since $E(\Delta_*)x + E(\Delta_0)x = E(\sigma(T))x = x$, we obtain $x = E(\Delta_0)x$, showing "$\Rightarrow$". □

In particular, if for $\lambda_0 \in \sigma(T)$ we take $f(\lambda) = \lambda - \lambda_0$, then $E(\{\lambda_0\})$ is the projection onto $\ker(T - \lambda_0 I)$, the eigenspace of $T$ corresponding to eigenvalue $\lambda_0$ if $E(\{\lambda_0\}) \neq 0$. *Isolated points of $\sigma(T)$ are eigenvalues of $T$* (as relatively open subsets of $\sigma(T)$, by part (iii) of Theorem 9.).□

There is another functional model for normal operators, very useful in many situations and considered as an alternative form of spectral theorem. We say that $f_0 \in H$ is a **cyclic vector** for an operator $T \in \mathcal{B}(H)$, if $H$ is the closed linear span of $\{T^n f_0 : n \in \mathbb{Z}_+\}$, where $T^0 f = f$. Equivalently, $H$ is the smallest closed linear subspace invariant for $T$. It is an easy exercise to verify that if $K \subset H$ is an invariant subspace for $T \in \mathcal{B}(H)$, then $(T|_K)^*$, the adjoint of its restriction (computed in $\mathcal{B}(K)$) equals $P_K T^*|_K$ and if $T^*x \notin K$, then $\|P_K T^* x\| < \|T^* x\|$.

Hence if $T$ is normal, then $T|_K$ is normal iff $K$ is a **reducing subspace for** $T$ (i.e. invariant both for $T$ and for $T^*$, or equivalently, both $K$ and $K^\perp$ are invariant for $T$). Given $f_0 \in H$ let $H_T(f_0)$ be the smallest reducing $T$ closed subspace of $H$ that contains $f_0$. Clearly, if $T$ is normal, this subspace is the closed linear span of $\{(T^*)^k T^n f_0 : k, n \in \mathbb{Z}_+\}$. If moreover $T$ is self-adjoint, then its invariant subspaces are reducing and then $f_0$ is cyclic, iff $H_T(f_0) = H$ and then $H$ is the closure of $\{p(T)f_0 : p \in \mathbb{C}[t]\}$. In case of normal operators $T$ we say that $H_T(f_0)$ is the star-cyclic subspace generated by $f_0$ and if this subspace equals $H$, wa say that $T$ is ***-cyclic, with *-cyclic vector $f_0$.**

**Theorem 12.** *A normal operator $T \in \mathcal{B}(H)$ is unitarily equivalent to a multiplication operator. In a special case when $T = T^*$ and $T$ possesses a cyclic vector, $T$ is unitarily equivalent to $M_t$ on $L^2(\sigma(T), \mu)$ for some probabilistic Borel measure $\mu$, where $M_t f(t) = tf(t)$ a.e.$[\mu]$ for $f \in L^2(\sigma(T), \mu)$.*

PROOF. We begin with the cyclic case. Without loss of generality, assume that $\|f_0\| = 1$. For a polynomial $p \in \mathbb{C}[t]$ let us define a (linear) functional $\Lambda(p) := \langle p(T)f_0|f_0\rangle$. Since $\sigma(p(T)) = p(\sigma(T))$ and $\|p(T)\| = r(p(T))$, we have $\|p(T)\| = \|p\|$ -the norm of $p$ in $C(\sigma(T))$. Hence $|\Lambda(p)| \leq \|f_0\|^2\|p\| = \|p\|$ and $\Lambda$ extends by continuity to $C(\sigma(T))$, where the (complex) polynomials form a dense subspace. This extension, denoted again by $\Lambda$, is positive. Indeed, it follows either by Property 5. of functionals, since $\|\Lambda\| = 1 = \Lambda(\mathbf{1})$, or directly: (if $u \in C(\sigma(T))$ is positive, $u = w^2$, where $w \geq 0$, then $\Lambda u = \langle w(T)|w(T)\rangle \geq 0$). By Riesz' representation Theorem, $\Lambda(v) = \int v \, d\mu$ for some (probabilistic) Borel measure $\mu$ concentrated on $\sigma(T)$. For a complex polynomial $p \in \mathbb{C}[t]$ let $U_0(p) := p(T)f_0$. The domain of $U_0$ is dense [11] in $L^2(\mu)$ and its range is dense in $H$, because $f_0$ is cyclic. Since

$$\|U_0 p\|^2 = \langle p(T)f_0|p(T)f_0\rangle = \langle (p(T))^* p(T)f_0|f_0\rangle = \langle |p|^2(T)f_0|f_0\rangle = \int |p|^2 \, d\mu,$$

this $U_0$ is an isometry, hence it extends to a unitary map $U : L^2(\mu) \to H$. Now, as $(M_t p)(T) = Tp(T)$, we have $U(M_t p) = Tp(T)f_0 = T(Up)$ for $p$ from a dense subspace $\mathbb{C}[t]$ of $L^2(\mu)$ and this implies that $UM_t = TU$, i.e. $T = UM_t U^*$.

For arbitrary self-adjoint operator $T$ we may find (using Kuratowski-Zorn Lemma) a maximal family of vectors $f_\iota \in H$ such that subspaces $H_\iota := H_T(f_\iota)$ (spanned by $\{p(T)f_\iota : p \in \mathbb{C}[t]\}$) are mutually orthogonal. Then since $H_\iota$ are invariant (and reducing) subspaces, $T$ can be represented as a (possibly infinite) orthogonal sum of its restrictions $T_\iota := T|_{H_\iota} \in \mathcal{B}(H_\iota)$. The $T_\iota$ are self-adjoint and have cyclic vectors (namely, $f_\iota$). Here we use the maximality assumption to check, that the span of $\bigcup_\iota H_\iota$ equals $H$ (otherwise the ortho-complement to this union would be invariant for $T$, hence containing another "cyclic subspace" for $T$). The spectra $\Omega_\iota := \sigma(T_\iota)$ are contained in $\sigma(T)$ and we obtain the corresponding measures $\mu_\iota$ on $\Omega_\iota$. On disjoint union $\Omega := \coprod_\iota \Omega_\iota = \bigcup_\iota \{\iota\} \times \Omega_\iota$ we consider

[11] That complex polynomials are dense (even uniformly, hence in $L^2$ -norm) in $C(K)$ when $K$ is a compact subset of $\mathbb{R}$ follows from Weierstrass' Approximation Theorem. The density of $C(K)$ in $L^2(K, \mu)$ for bounded Borel measures follows from regularity of such measures on compact metric spaces.

the direct sum of Borel sigma-algebras, so $\Delta \subset \Omega$ is measurable iff $\forall_\iota\ \Delta \cap (\{\iota\} \times \Omega_\iota)$ is Borel-measurable and the measure $\mu(\Delta) = \sum_\iota \mu_\iota(\{\omega \in \Omega_\iota : (\iota, \omega \in \Delta \cap (\{\iota\} \times \Omega_\iota)\})$. Then it is a routine exercise to verify that $T$ is unitarily equivalent to $M_\varphi$ on $L^2(\Omega, \mu)$, where for $x = (\iota, t) \in \{\iota\} \times \Omega_\iota$ we define $\varphi(x) := t$. One can prove (using Exercise formulated below) that if $H$ is separable, then such sums are at most countable and if the indices $\iota$ run through $\mathbb{N}$, then by replacing $\mu_\iota$ with $\tilde{\mu}_\iota := 2^{-\iota}\mu_\iota$ we obtain unitarily equivalent multiplication operators $M_\varphi$ on $L^2(\tilde{\mu})$ with a bounded measure $\mu$.

The proof so far wasn't using Spectral Theorem. For normal *-cyclic operators (so that $H = H_T(f_0)$ for some normalised vector $f_0$) one shows directly that $\mu(\Delta) := \mu_{f_0}(\Delta) := \langle E(\Delta)f_0|f_0\rangle$ gives us a probabilistic Borel measure on $\sigma(T)$ such that $T$ is equivalent to $M_\lambda \in \mathcal{B}(L^2(\mu))$, the multiplication by identity function on $\sigma(T)$. Here $E$ is the spectral measure of $T$. Applying Proposition 7 with $\mathfrak{M}$ equal the Borel sigma-field on $\sigma(T)$, for $\phi \in L^\infty(\mathfrak{M})$ - a bounded Borel function denoting $\phi(T) := \int \phi\, dE$ we have

$$\|\phi(T)f_0\|^2 = \langle \phi(T)f_0|\phi(T)f_0\rangle = \langle |\phi|^2(T)f_0|f_0\rangle = \int |\phi|^2 d\mu,$$

which means that the mapping $L^\infty(\mathfrak{M}) \ni \phi \mapsto \phi(T)f_0 \in H$ is an isometry with respect to the $L^2(\mu)$ -norm. By density (of equivalence classes of $\phi \in L^\infty(\mathfrak{M})$ in $L^2(\mu)$ , its continuous extension $V$ to an isometry from $L^2(\mu)$ onto $H$ is a unitary map. If $M_z : L^2(\mu) \ni f(z) \mapsto zf(z) \in L^2(\mu)$ is the multiplication by identity on $\sigma(T)$, then $VM_zV^{-1}(\phi(T)f_0) = VM_z\phi = V(z\phi(z)) = T\phi(T)f_0$. Hence $VM_zV^{-1} = T$ on vectors of the form $\phi(T)f_0$ for arbitrary $\phi \in L^\infty(\mathfrak{M})$ -which are dense in $H$ (by *-cyclicity). Hence $VM_zV^{-1} = T$.

In general case we proceed as in the self-adjoint case, decomposing $T$ into a direct orthogonal sum of *-cyclic normal operators and we use a disjoint union of the corresponding measure spaces as above. □

EXERCISE. If $\mu$ and $\nu$ are mutually absolutely continuous finite measures, say $d\nu(t) = \rho(t)\, d\mu(t)$ with Radon-Nikodym derivative $\rho > 0\,[\mu]$ -a.e., then the mapping $U : L^2(\nu) \ni f(t) \mapsto f(t)\sqrt{\rho(t)} \in L^2(\mu)$ is unitary and it establishes equivalence between multiplication operators $M_\varphi$ acting on both spaces, when $\varphi \in L^\infty(\mu)$. Note that $L^\infty(\mu) = L^\infty(\nu)$ in this case.

EXERCISE. If $M_\phi \in \mathcal{B}(L^2(X, \mathfrak{M}, \mu))$ is a multiplication operator, its polar decomposition can be obtained directly by decomposing the function $\phi$ point-wise: $\phi(x) = \text{sign}\,(\phi(x)) \cdot |\phi(x)|$. Here sign sends $re^{it}$ to $e^{it}$ if $r > 0$ and $\text{sign}(0) = 0$ This corresponds to general polar decomposition of an operator: $T = V|T|$, where $|T| := (T^*T)^{\frac{1}{2}}$ and $V$ is a partial isometry (i.e. the restriction of $V$ to $\ker(V)^\perp$ -the ortho-complement of its nullspace is an isometry). Check that for a self-adjoint operator $T$ the partial isometry from its polar decomposition is a difference of two orthogonal projections.

EXERCISE. Let $T \in \mathcal{B}(H)$. Since for any $f \in H$ we have $\|Tf\| = \|\,|T|f\,\|$, the assignment $V : |T|f \mapsto Tf$ is well-defined and $T = V|T|$. Check that $V$ is indeed a partial isometry and that $V^*V$ is a projection. Conversely, if $V^*V$ is a projection, then $V$ is a partial isometry.

EXERCISE. Is it true that $|T| = |T^*|$? (Consider the unilateral shift $S$.) Find the polar decompositions of $S$ and of $S^*$.

# 3 Compact operators

Let us recall a notation $u \otimes v$ or $u \otimes \overline{v}$ for the rank-one operator defined for $f \in H$ by

$$(u \otimes \overline{v})f := \langle f|v\rangle u, \quad \text{where } u, v \in H \setminus \{0\}$$

with the following properties: $\|u\otimes\overline{v}\| = \|u\|\|v\|$, $(u\otimes\overline{v})^* = v\otimes\overline{u}$, $w\otimes\overline{w} \geq 0$ and $A(u\otimes\overline{v})B = (Au)\otimes\overline{B^*v}$ for $A, B \in \mathcal{B}(H)$. A finite-rank[12] linear operator $T : H \to H$ (here we assume that $T \neq 0$) is continuous iff $\ker(T)$ is closed and in this case there exist $k(=\text{rk}(T) < \infty)$ and vectors $u_1, \ldots, u_k, v_1, \ldots, v_k \in H$ such that $T = \sum_{n=1}^k u_n\otimes\overline{v_n}$. Such operators are compact and (in Hilbert spaces) any compact operator is a limit of continuous finite-rank operators. (Recall that a linear operator $T$ is compact, if the closure of its image of any bounded set is compact, equivalently, images under $T$ of any bounded sequence contain norm-convergent subsequences. In Hilbert (or in reflexive) spaces equivalent to compactness is the property of carrying weakly-convergent[13] sequences $(x_n \rightharpoonup x)$ into norm convergent sequences: $\|Tx_n - Tx\| \to 0$. The set of compact operators $T : H \to H$, denoted by $\mathcal{B}_0(H)$ is a closed 2-sided ideal in $\mathcal{B}(H)$.)

EXERCISE 3.1 Show that $T \in \mathcal{B}_0(H) \Leftrightarrow (\|Tx_n\| \to 0$ whenever $x_n \rightharpoonup 0)$. Then show that $T \in \mathcal{B}_0(H) \Leftrightarrow |T| \in \mathcal{B}_0(H) \Leftrightarrow T^*T \in \mathcal{B}_0(H)$.

[12] The cardinal number $\text{rk}(T) := \dim T(H)$ is called the rank of $T$

[13] Since closed balls in $H$ are weakly compact and in separable spaces they are metrisable, the bounded sequences have weakly convergent subsequences. (Note that the closed linear span of a sequence is separable and can replace $H$.)

EXERCISE 3.2 Show that any non-zero eigenvalue $\lambda$ of $T \in \mathcal{B}_0(H)$ satisfies $\dim(\ker(\lambda I - T)) < \infty$.
EXERCISE 3.3 Find the "modulus" : $|u \otimes \overline{v}|$ and polar decomposition of rank-one operator. Find the spectral measure and the spectrum of a self-adjoint rank-one operator $w \otimes \overline{w}$.
EXERCISE.3.4 If $S_n \to S$ in strong operator topology, i.e. $\forall_{x \in H} \|S_n x - Sx\| \to 0$, then for any $T \in \mathcal{B}_0(H)$ we have $\|S_n T - ST\| \to 0$ as $n \to \infty$.

From the formula for products it follows that (continuous) finite rank operators form a 2-sided ideal (denote it $\mathcal{B}_{00}(H)$ in the algebra $\mathcal{B}(H)$. Its closure is the ideal $\mathcal{B}_0(H)$. We are going to show that $\mathcal{B}_0(H)$ is the only norm-closed ideal in $\mathcal{B}(H)$, if $H$ is a separable Hilbert space. We begin with observing that $u \otimes \bar{v}$ is the only linear operator that transforms $u$ to the vector $\|u\|^2 v$ and vanishes on all vectors orthogonal to $u$. Using this we prove the following

**Proposition 13.** *Any (2-sided) ideal $J \neq \{0\}$ in $\mathcal{B}(H)$ should contain $\mathcal{B}_{00}(H)$.*

*Proof.* There exist $T \neq 0, T \in I$, hence for some $f, g \in H \setminus \{0\}$ $Tf = g$. Given arbitrary nonzero $u, v \in H$ it suffices to find $A, B \in \mathcal{B}(H)$ such that $BTA = c^2 \cdot u \otimes \bar{v}$ for some $c > 0$. Let $A = f \otimes \bar{u}$, $B = v \otimes \bar{g}$. Then $Ah = 0$ for $h \perp u$ and $Au = \|u\|^2 f$, hence $TAu = \|u\|^2 g$ and $BTAu = c^2 v$ for $c = \|u\|\|g\|$, while $BTAh = 0$ if $h \perp u$. Hence $c^{-2} BTA = u \otimes \bar{v} \in J$. □

The next two results are corollaries of spectral theorem.

**Proposition 14.** If $E$ is the spectral measure for a normal operator $N = \int z E(dz)$, then $N$ is compact iff for any $\delta > 0$ the projection $P_\delta := E(\{z \in \mathbb{C} : |z| > \delta\}$ has finite rank.

*Proof.* If $\chi_\delta$ is the characteristic function of $\{z \in \mathbb{C} : |z| > \delta\}$, then $N - NP_\delta = \int (z - z \cdot \chi_\delta) E(dz)$ has norm estimated by the sup-norm over $\sigma(N)$ of the function under integral sign, which is $\leq \delta$. Since $\delta > 0$ is arbitrary, $N$ is approximated in operator norm by finite-rank operator, hence is compact. □

**Theorem 15.** *Assume that $H$ is separable. If an ideal $J$ in $\mathcal{B}(H)$ contains a non-compact operator $T$, then $J = \mathcal{B}(H)$. Since any proper ideal in $\mathcal{B}(H)$ contains $\mathcal{B}_{00}(H)$, $\mathcal{B}_0(H)$ is the only norm-closed proper ideal in $\mathcal{B}(H)$.*

*Proof.* Then the self-adjoint operator $T^*T = \int_{[0,M]} tE(dt)$ belongs to $J$ (where $M = \|T\|^2$) and for some $\delta > 0$ the rank of $P_\delta := E([\delta, M])$ is infinite. For $\chi_\delta$ as in the previous proof, by multiplicative property of the spectral integral,

$$P_\delta = \int t^{-1} \chi_\delta \, E(dt) \int t \, E(dt) = \Big( \int t^{-1} \chi_\delta \, E(dt) \Big) T^*T \in J,$$

since the first integral defines a bounded linear operator. Now, since $\dim P_\delta(H) = \aleph_0 = \dim H$, there exists a surjective isometry $V : H \to P_\delta(H)$ and then the identity operator $I_H$, equal $V^* P_\delta V$ belongs to $J$, so $J = \mathcal{B}(H)$. The last claim follows, since $\mathcal{B}_{00}(H)$ is dense in $\mathcal{B}_0(H)$. □

EXERCISE 3.5 If $\lambda \in \sigma_{ap}(T) \setminus \{0\}$, where $T \in \mathcal{B}_0(H)$, then $\lambda$ is an eigenvalue of $T$ and $\dim \ker(T - \lambda I) < \infty$.
EXERCISE 3.6 If $S \in \mathcal{B}(H)$ is injective, then its range: $S(H)$ is closed iff $S$ is bounded below, i.e. if $\inf\{\|Sx\| : \|x\| = 1\} > 0$.

**Theorem 16.** *The non-zero part of $\sigma(T)$ for a compact operator $T$ consists of eigenvalues and for $\lambda \neq 0$ the range of $T_\lambda := T - \lambda I$ is closed.*

Proof. First we note that if $\lambda \notin \sigma_p(T), \lambda \neq 0$ and $\bar{\lambda} \notin \sigma_p(T^*)$, then $\lambda \notin \sigma(T)$. Indeed, by Exercise 3.5, $\lambda \notin \sigma_{ap}(T)$, so $T_\lambda := T - \lambda I$ is bounded from below. Hence it has a closed range by Exercise 3.6. But $\{0\} = \ker((T_\lambda)^*)$ is the ortho-complement of the range of $T_\lambda$, showing the surjectivity of $T_\lambda$. Therefore it suffices to show assuming that $\lambda \notin \sigma_p(T)$, that the range of $T_\lambda$ is $H$ (so that $\lambda \notin \sigma(T)$). Assume the contrary: let $X_1 := T_\lambda(H)$ be a proper (closed) subspace of $H$. Since $(T_\lambda)^n$ is bounded below for any $n \in \mathbb{N}$, we have a strictly decreasing sequence of closed subspaces $X_n := (T_\lambda)^n(H)$. (Indeed, if $y_0 \in H \setminus X_1$, then $(T_\lambda)^n y_0 \in X_n \setminus X_{n+1}$, since $X_{n+1} = (T_\lambda)^n(X_1)$ and $(T_\lambda)^n$ is injective.) Hence we can find a sequence of vectors $y_n \in X_n$ such that $\|y_n\| = 1$ and $y_n \perp X_{n+1}$. Then for $k > n$

$$\|Ty_n - Ty_k\|^2 = \|T_\lambda y_n - T_\lambda y_k + \lambda(y_n - y_k)\|^2 = \|\lambda y_n\|^2 + \|R\|^2 \geq |\lambda|^2,$$

where $R = T_\lambda(y_n - y_k) + \lambda y_k \in X_{n+1} \perp y_n$. Therefore no subsequence of $(Ty_n)$ can converge, as it fails Cauchy's condition, contradicting the compactness of $T$.

It remains to show that $T_\lambda(H)$ is closed for $\lambda \in \sigma(T)$ (in the remaining case this is obvious). Since we have the decomposition $H = \ker(T_\lambda) \oplus Y_\lambda$, where $Y_\lambda$ is the ortho-complement of $\ker(T_\lambda)$, the range of $T_\lambda$ equals the range: $S(Y_\lambda)$ of its restriction: $S := T_\lambda|_{Y_\lambda}$. In view of Exercise 3.6 it remains to show that $S$ is bounded below. Assume the contrary: that for a sequence of vectors $x_n \in Y_\lambda$ with $\|x_n\| = 1$ we have $Sx_n = Tx_n - \lambda x_n \to 0$. By compactness, after passing to a subsequence, if necessary, $Tx_n$ converges to some $y_0$, hence also $\lambda x_n \to y_0$ and $\|\lim x_n\| = \lim \|x_n\| = 1$. By continuity, $S(\lim x_n) = 0$ and since $S$ is injective, $\lim x_n = 0$, a contradiction. □

**Remark.** As a corollary we obtain the **Fredholm Alternative**: *If $T \in \mathcal{B}_0(H)$ and $\lambda \neq 0$, then either the equation $Tx - \lambda x = y$ has exactly one solution for any $y \in H$ or the equation $Tx - \lambda x = 0$*

*has non-zero solutions.* The above proof carries over to Banach spaces, but instead of orthogonality one has to apply Riesz' Lemma and in the final part one takes as $X_\lambda$ some closed subspace forming $H$ as a direct sum with $\ker(T_\lambda)$. One can also show that for any non-zero $\lambda$ the dimensions of $\ker(T_\lambda)$ and of $\ker(T_\lambda)^*$ are equal.

A lesser known[14], but useful are the following criteria of compactness for $T \in \mathcal{B}(H)$

**Theorem 17.**(a) $T$ is compact iff for any orthonormal sequence $(e_n)$ we have $\|Te_n\| \to 0$ as $n \to \infty$.
(b) $T$ is compact iff its range does not contain a closed infinite-dimensional subspace.
(c) $T$ is compact iff its range $T(H)$ is contained in a range of some compact operator.

## 3.1 Calkin algebra

Since $\mathcal{B}_0 := \mathcal{B}_0(H)$ is a 2-sided ideal in $\mathcal{B}(H)$ such that $T^* \in \mathcal{B}_0$ for any $T \in \mathcal{B}_0$, the quotient space $\mathcal{B}(H)/_{\mathcal{B}_0}$ is a C*-algebra (with natural algebraic operations defined to make the canonical surjection $\pi : T \mapsto [T] := T + \mathcal{B}_0$ a *-preserving homomorphism. This quotient algebra is called the **Calkin algebra**.

In non-commutative unital C*-algebras $A$ one considers also **left-invertible** (resp. right-invertible) **elements** $x$. As in the case of $G_A$-the set of invertible elements, these are open subsets of $A$ and we have analogous concepts of **left-spectrum:**

$$\sigma_\ell(x) := \{\lambda \in \mathbb{C} : \forall_{a \in A} a(\lambda \mathbf{1} - x) \neq \mathbf{1}\}$$

and the corresponding right-spectrum $\sigma_r(x)$, where $(\lambda \mathbf{1} - x)b \neq \mathbf{1}$ for all $b \in A$. Clearly, $\sigma(x) = \sigma_\ell(x) \cup \sigma_r(x)$. For normal elements these 3 types of spectra coincide, as follows from the following $a \in C^*(x, \mathbf{1})$

**Lemma 18.** *If $x$ is normal, then $x$ is left-invertible iff $x$ is right -invertible (and then the left- and right- inverses of $x$ are unique and coincide with $x^{-1}$.*

*Proof.* Assume that $0 \notin \sigma_\ell(x)$, so that for some $z \in A$ we have $zx = \mathbf{1}$. Then for any we have $\|a\| = \|zxa\| \leq \|z\|\|xa\|$, hence

$$\|xa\| \geq \delta \|a\| \quad \text{where} \quad \delta := \|z\|^{-1} > 0, ,$$

which implies that $0 \notin \sigma(x)$. In fact, assuming the opposite and using the C*- isomorphism between $a \in C^*(x, \mathbf{1})$ and $C(\sigma(x))$ carrying $x$ to the identity function $(= \hat{x})$ on $\sigma(x)$ (containing 0), one chooses $a \in C^*(x, \mathbf{1})$ equal $\|a\|$ at 0 and vanishing at $\{\lambda \in \sigma(x) : |\lambda \geq \frac{\delta}{2}$. Then $\|xa\| \leq \frac{\delta}{2}$, a contradiction with $\|xa\| \geq \delta\|a\|$. Applying this proof to $x^*$ in place of $x$ we show the remaining claim, since $0 \notin \sigma_\ell(x^*) \Leftrightarrow 0 \notin \sigma_r(x)$. □

**Remark.** In $\mathcal{B}(H)$ an operator $T$ is left-invertible iff $\inf\{\|Tx\| : \|x\| = 1\} > 0$, i.e. $T$ is bounded from below. Indeed, if $ST = I$, for $x \in H$ we have $\|x\| = \|STx\| \leq \|S\|\|Tx\|$. Hence $\sigma_\ell(T)$ is the approximate point spectrum, $\sigma_a(T)$. (Find analogous description of the right spectrum of operators).

**Definition** If $\pi : \mathcal{B}(H) \to \mathcal{B}(H)/_{\mathcal{B}_0}$ is the canonical surjection, the **essential spectrum** $\sigma_e(T)$ is the spectrum of $\pi(T)$ in the Calkin algebra. Analogously, we define the **left-essential** (**right essential**) spectra of $T$ as $\sigma_{\ell e}(T) := \sigma_\ell(\pi(T))$ and $\sigma_{re}(T) := \sigma_r(\pi(T))$ -respectively. We say that $T \in \mathcal{B}(H)$ is a **Fredholm operator** if $\pi(T)$ is invertible in $\mathcal{B}(H)/_{\mathcal{B}_0}$. Then we write $T \in \mathcal{F}(H)$ The sets $\mathcal{F}_\ell(H)$, $\mathcal{F}_r(H)$ of **left-** resp. **right-Fredholm** operators are defined analogously by one-sided invertibility of $\pi(T)$. An operator is **semi-Fredholm**, if it is either right- or left-Fredholm.

One can prove (Atkinsons's Theorem) that $T \in \mathcal{B}(H)$ is Fredholm iff its range $ran(T)$ is closed, has finite co-dimension $(:= \dim(ran(T)^\perp)$ and $\dim(\ker(T)) < \infty$

**Theorem 19.** For $T \in \mathcal{B}(H), \lambda \in \mathbb{C}$, if $T_\lambda := T - \lambda I$, the following are equivalent

1. $\lambda \notin \sigma_{\ell e}(T)$ (i.e. $T_\lambda$ is a left-Fredholm operator.
2. $T_\lambda(H)$ is closed and $\dim(\ker(T_\lambda)) < \infty$.
3. If $h_n \rightharpoonup 0$ and $\|h_n\| = 1$ then $\liminf \|T_\lambda h_n\| > 0$.
4. There is no orthonormal sequence $(e_n)$ with $\lim \|T_\lambda e_n\| = 0$.
5. $\exists_{\delta > 0}$ such that $\{x \in H : \|T_\lambda h\| \leq \delta \|h\|\}$ contains no infinite-dimensional subspace.
6. For $E =$ the spectral measure of $|T_\lambda| := (T_\lambda^* T_\lambda)^{\frac{1}{2}}$ and for some $\delta > 0$ $\dim E([0, \delta]) < \infty$.
7. $\forall_{K \in \mathcal{B}_0(H)}$ we have $\dim \ker(T_\lambda + K) < \infty$.

[14]These results are contained in the paper: P.A.Fillmore, J.P. Williams "On operator ranges", Advances in Math. 7 (1971), 254-281.

For the proof see Chapter XI of [1], or [3]

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