Fundamental Properties and Theorems of Operator Calculus Based on the Laplace Transform

1. *Linearity Theorem*

$$\mathcal{L}[\alpha_1 f_1(t)+\alpha_2 f_2(t)]=\alpha_1\mathcal{L}[f_1(t)]+\alpha_2\mathcal{L}[f_2(t)]$$

2. *Transformation of Function Derivatives*

$$\mathcal{L}[f^{(n)}(t)]=s^n\cdot F(s)-s^{n-1}\cdot f(0^+)-s^{n-2}\cdot f'(0^+)-\ldots-s\cdot f^{(n-2)}(0^+)-f^{(n-1)}(0^+)$$

$$\mathcal{L}[\ddot{f}(t)]=s^2\cdot F(s)-s\cdot f(0^+)-\dot{f}(0^+)$$

$$\mathcal{L}[\dot{f}(t)]=s\cdot F(s)-f(0^+)$$

3. *Transformation of Function Integral*

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right]=\frac{1}{s}\cdot F(s)$$

4. *Shift in the Real Domain (Time-Shift)*

$$\mathcal{L}[f(t-\tau)]=e^{-\tau s}\cdot F(s)$$

5. *Shift in the Complex Domain (Frequency-Shift)*

$$\mathcal{L}[e^{-at}\cdot f(t)]=F(s+a)$$

6. *Scaling Theorem*

$$\mathcal{L}[f(a\cdot t)]=\frac{1}{a}\cdot F\left(\frac{s}{a}\right)$$

7. *Complex Domain Differentiation*

$$\mathcal{L}[t\cdot f(t)]=(-1)\frac{dF(s)}{ds}$$

$$\mathcal{L}[t^n\cdot f(t)]=(-1)^n\frac{d^nF(s)}{ds^n}$$

8. *Final Value Theorem*

If exists: $\lim_{t\to\infty} f(t)$ *and* $\mathcal{L}[f(t)]=F(s)$ *then:*

$$\lim_{t\to\infty} f(t)=\lim_{s\to 0^+} s\cdot F(s)$$

9. *Convolution Theorem*

$$\mathcal{L}[f_1(t) * f_2(t)] = \mathcal{L}[f_1(t)] \cdot \mathcal{L}[f_2(t)] = F_1(s) \cdot F_2(s),$$

where the convolution of functions $f_1(t)$, $f_2(t)$.

is defined by the relation:

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) \cdot f_2(t-\tau) d\tau$$

Definitions

Let f(t) be a function of real t (i.e. time) that satisfies the following conditions:

1) f(t) = 0 for t < 0,

2) f(t) satisfies Dirichlet conditions:
   - The interval in which the function is defined can be divided into open subintervals where f(t) is monotonic.
   - At each point, the following holds: $f(t) = [f(t^-) + f(t^+)] / 2$.

3) f(t) is integrable over every finite interval, and the following inequality holds:
   $|f(t)| \leq Me^{at}$ (M>0, a>0, for $t>t_0$).

Laplace Transform (for complex s, Re(s)>a):

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

Inverse Laplace Transform:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\alpha - j\infty}^{\alpha + j\infty} F(s) e^{st} ds$$

Laplace Transforms of Frequently Used Functions

Original f(t)	Transform F(s)
δ(t) – Dirac impulse	1
1(t) – unit step (Heaviside fcn)	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
$e^{\mp at}$	$\frac{1}{s \pm a}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$