Simulation of deterministic chaos

Outline

- chaos dynamics in the past
- some definitions - glossary
- iterative maps
    - 1D logistic map
    - bifurcations, period doubling, Feigenbaum constant
    - 2D Henon maps
- dumped driven pendulum
    - phase diagram
    - Poincare map
    - Lyapunov exponent
- Lorenz equations

Simulation of deterministic chaos

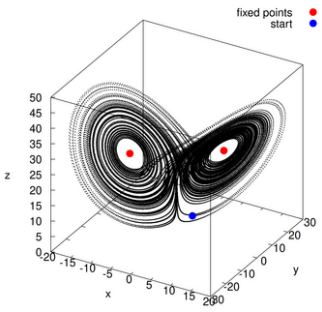
When did the people encounter the chaotic motion?

- Newton noticed that heavy-mass Jupiter ($M_{JUPITER} = 318\ M_{EARTH}$) must perturb the orbital motion of the Earth, he wonder if this fact could make the Earth orbit unstable or chaotic - **three body problem**

- at the early of 20 century **Pierre Fatou** and **Gaston Julia** introduced the concept of sensitivity to initial conditions,

- **chaotic dynamic is observed when nonlinear terms appear in the equations of motion**:
  - anharmonic oscillator – Georg Duffing (1918)
  - oscillations in electric circuits – Balthasar van der Pol (1927)

- in 1961 mathematician/meteorologist **Eward Lorenz** investigating the solutions of simplified Navier-Stokes equations accidently discovered that even a slight change in the initial conditions may generate completely different trajectories - **the butterfly effect**



Does the flap of the butterfly's wings in Brazil set off a tornado in Texas?

(originally, it was a seagull)

the answer is: NO - we didn't observe a tornado crossing the equator

- generally the chaotic dynamics arises due to nonlinearity terms in ODE systems, till now it has been found in many scientific fields: physics, chemistry, biology, mathematics, computer science

Simulation of deterministic chaos

- **dynamical system** – system that is modelled by a set of the differential equations and the initial conditions
  (pendulum, atomic gas in molecular dynamics)

- **non-linear dynamics** – the differential equations include the non-linear terms with respect to
  positions, velocities or accelerations

- **deterministic system** – the dynamical system that evolves along single unique trajectory
  for given set of initial conditions (**trajectories can not cross each other**)

- **fixed point** – the solution is identical to some set of the initial conditions,
  in other words, the specific solution which is constant in time
  (system stays in mechanical equlibrium),
  remember: for some other sets of initial conditions,
  the trajectories may evolve toward
  the same fixed point asymptotically

- **orbit** – set of points in the phase space that satisfy the set of differential equations
  goverining the dynamics of the system

- **periodic orbit** – the set of points in phase space which are repeatedly visited by a system

- **limit cycle** – a closed curve in phase space toward which the orbit moves over the time
  (driven weakly-dumped pendulum)

- **attractor** – the set of points toward which the trajectory of a system tends in phase space over the time,
  can be a single fixed point, a limit cycle or a set of points without a simple functional connection

- **chaotic attractor** – it is an attractor for a nonlinear dynamical system trajectory,
  when the solution strongly depends on the small perturbations in initial conditions,
  usually it has a complicated geometrical structure,
  the dynamics of such system is unpredictable (chaotic)

Iterative map, logistic map

The ODE system can be solved numerically with one-step method, the solution is then found iteratively

$$\frac{d\vec{u}}{dt} = f(t, \vec{u}) \quad \Longrightarrow \quad \vec{u}_{i+1} = F(t_i, \vec{u}_i)$$

this iterative map allows for solving the problem without referring to the differential calculus

As an example let's consider simple Malthusian problem (Verhulst, 1838) of **population growth**

$$\frac{du}{dt} = ru - \frac{r}{k}u^2$$

r – reproduction rate (growth)
k - carrying capacity (population decline)

Analysis of population stability gives

$$\frac{du}{dt} = 0 \quad \Longrightarrow \quad ru - \frac{r}{k}u^2 = 0 \quad \Longrightarrow \quad u = k$$

The population size grows until it achieves the maximal environmental capacity.

The simplest discretized iterative map we get for Forward Euler scheme

$$\frac{du}{dt} = ru - \frac{r}{k}u^2 \quad \Rightarrow \quad u_{i+1} = u_i + \Delta t\, f(t_i, u_i)$$

$$u_{i+1} = u_i + \Delta t \left(ru_i - \frac{r}{k}u_i^2 \right) = u_i \left(1 + \Delta tr - \frac{\Delta tr}{k}u_i \right)$$

substitution: $\lambda = 1 + \Delta tr$

$$u_{i+1} = \lambda u_i \left(1 - \frac{\Delta tr}{\lambda k}u_i \right)$$

substitution: $\beta = \dfrac{\Delta tr}{\lambda k}$

$$\beta u_{i+1} = \lambda \beta u_i \left(1 - \beta u_i \right)$$

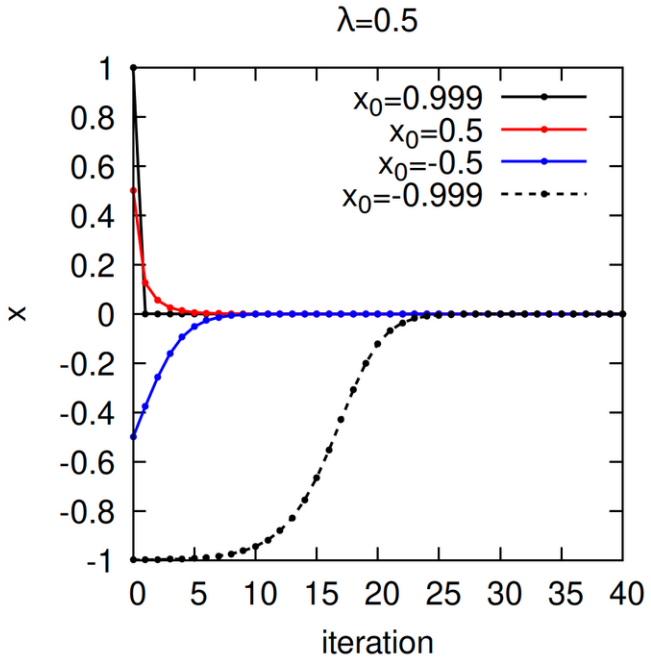
substitution: $x_i = \beta u_i$

we arrive at the **logistic map (equation)**

$$x_{i+1} = \lambda x_i (1 - x_i)$$

Now we forget about its physical origin and just focus on its numerical solutions...

The first solution we find for λ=0.5 numerically



- all trajectories converge to one fixed point x=0
- this fixed point looks like it attracts the trajectories
- the set of states (not necessary single points) the trajectories evolves to are called **attractors**
- occurence of attractors is an antithesis of ergodic (Hamilton) dynamics (the system shall evolve by sampling the phase space uniformly – here it doesn't)
- simulations have revealed one attractor, do there exist the others?

condition:

$$x_{i+1} = x_i \quad \Longrightarrow \quad x_{fixed} = \lambda x_{fixed}(1 - x_{fixed})$$

← quadratic equation (two roots)

$$x_{fixed} = 0 \quad \vee \quad x_{fixed} = 1 - \frac{1}{\lambda}$$

this we have found

this depends on λ

For the second fixed point we may estimate the range of its allowed values.
The analysis must rely on a behaviour of a trajectory close to this point

$$x_{i+1} = F(x_i) = F(x_{fixed} + \delta_i) = x_{fixed} + \delta_{i+1}$$

In vicinity of a fixed point we may expand the trajectory into series and keep only linear term

$$x_{i+1} = \underbrace{F(x_{fixed})}_{x_{fixed}} + \left.\frac{\partial F}{\partial x}\right|_{x=x_{fixed}} \cdot \delta_i = x_{fixed} + \delta_{i+1}$$

From comparison of both sides we get the condition for motion towards the x_{fixed}
(smaller and smaller spatial displacements δ)

$$\left|\frac{\delta_{i+1}}{\delta_i}\right| = \left|\left.\frac{\partial F}{\partial x}\right|_{x=x_{fixed}}\right| < 1$$

$$F(x) = \lambda x(1 - x^2)$$

$$\left.\frac{\partial F}{\partial x}\right| x = x_{fixed} = \lambda(1 - 2x_{fixed})$$

(1-st) $x_{fixed} = 0$

$$\left|\frac{\partial F}{\partial x}\right| = |\lambda| < 1$$

(2-nd) $x_{fixed} = 1 - \frac{1}{\lambda}$

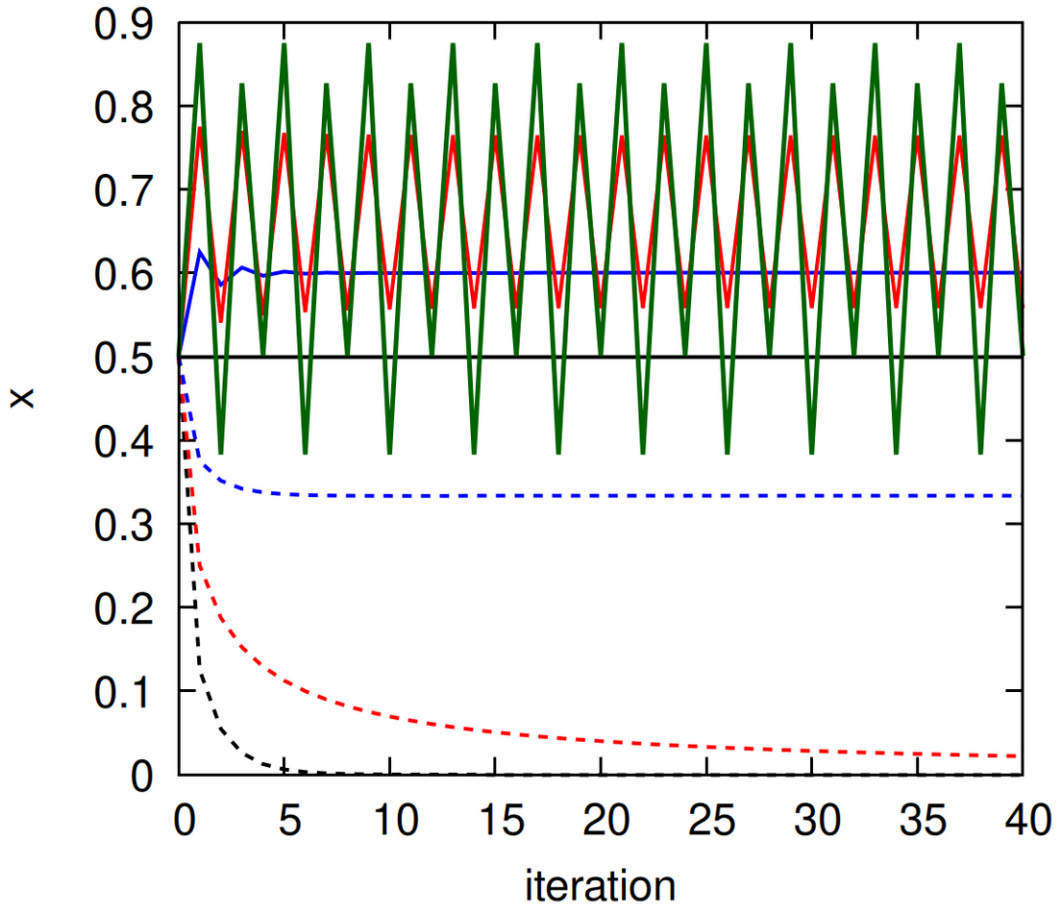
$$\left|\frac{\partial F}{\partial x}\right| = |2 - \lambda| < 1 \qquad \lambda \in (1, 3)$$

We get two fixed points and two ranges of parameter λ

$$x_{fixed} = 0 \iff \lambda \in (-1, 1)$$

$$x_{fixed} = 1 - \frac{1}{\lambda} \iff \lambda \in (1, 3)$$

But what happens when λ is larger?



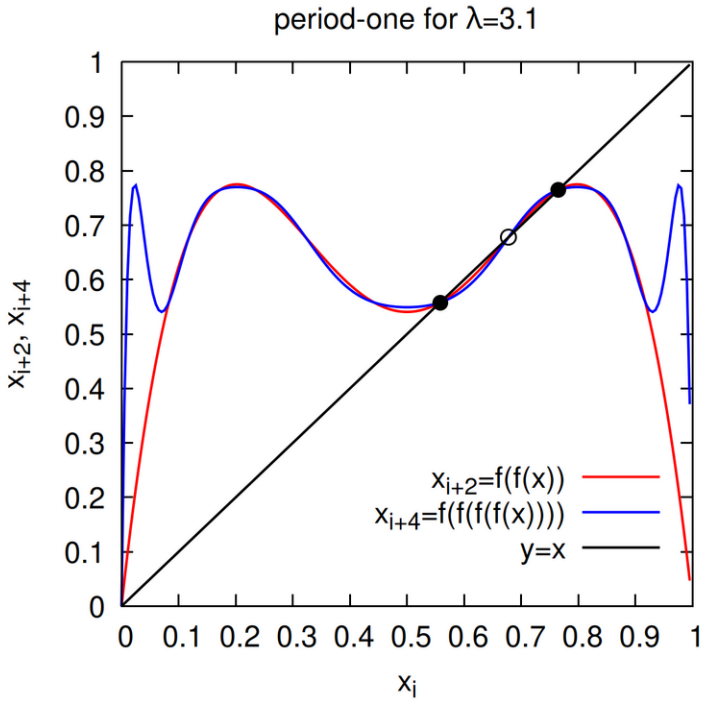
λ=0.5
λ=1.0
λ=1.5
λ=2.0
λ=2.5
λ=3.1
λ=3.5

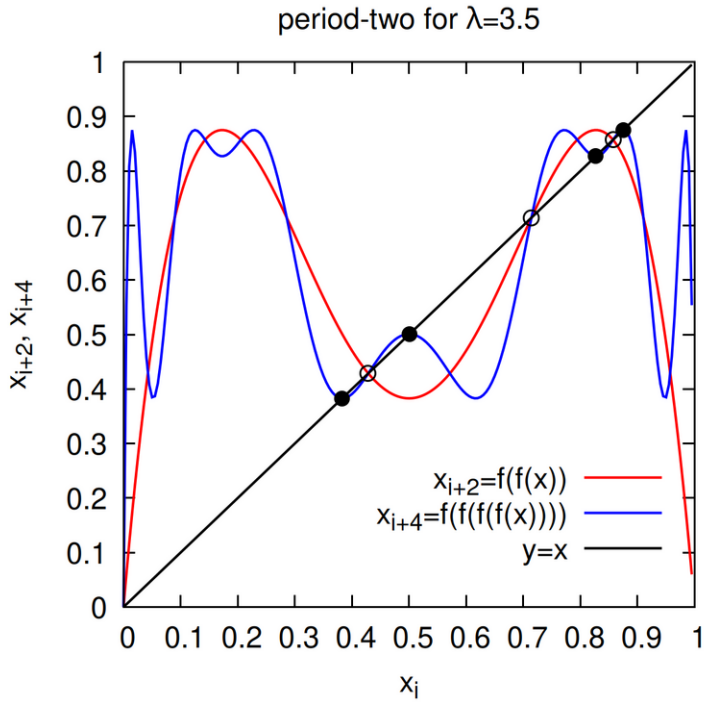
$\leftarrow x_{fixed}(\lambda = 3) = 0.666\ldots$

- when λ<3 the solution smoothly converges to a fixed point
- for larger values trajectory oscillates between:
  - two points (period-one)
  - four points (period-two)

- for λ>3 we get the **period-one** and **period-two** oscillations with **fixed turning points**
- periodicity in the iterative map means that after a few iterations we return to the initial point
- therefore we may analyze the relations between x_i, x_{i+2} and x_{i+4}

full dots – stable turning points
empty dots – unstable turning points



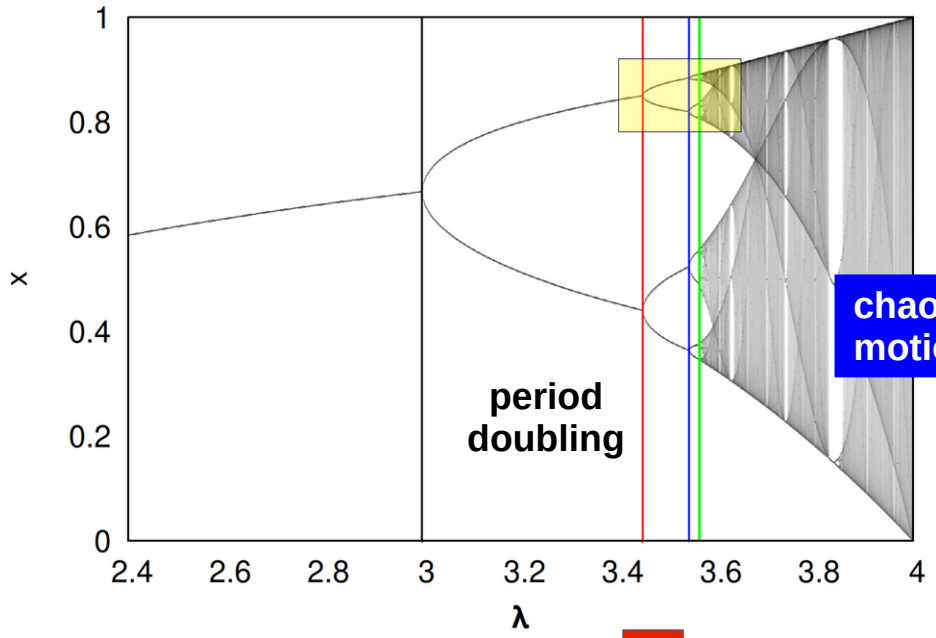


- for period-one solution we get dependence: $x_{i+4}=x_{i+2}=x_i$
- for period-two : $x_{i+4}=x_i$

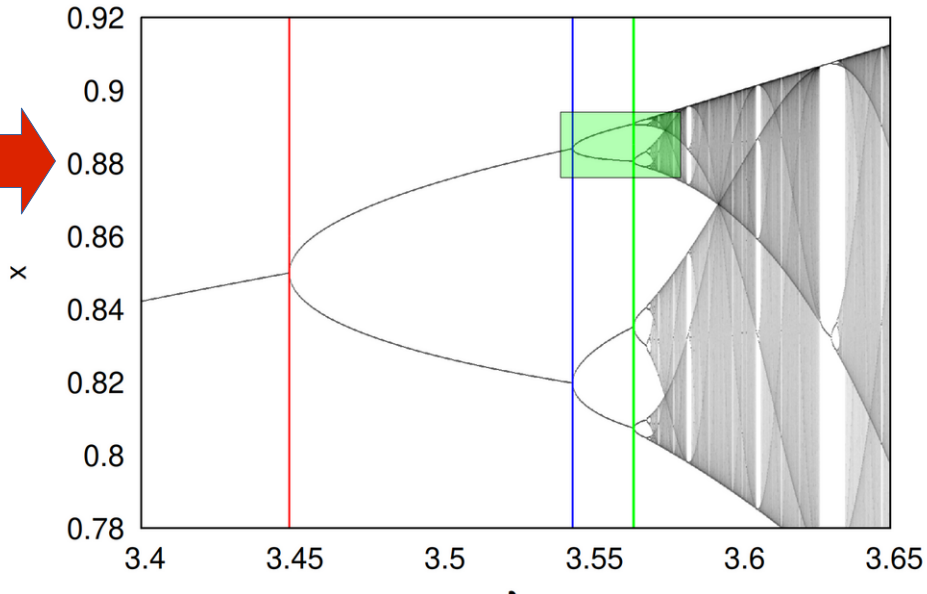
What happens for even larger λ? - this must be checked in more systematic way – full scan over λ range.

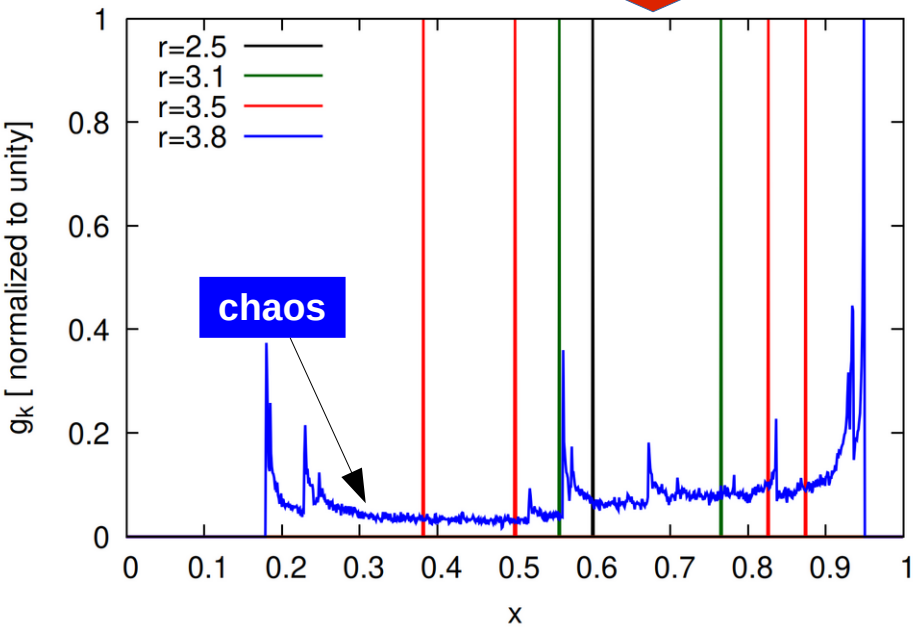
Simulation of deterministic chaos

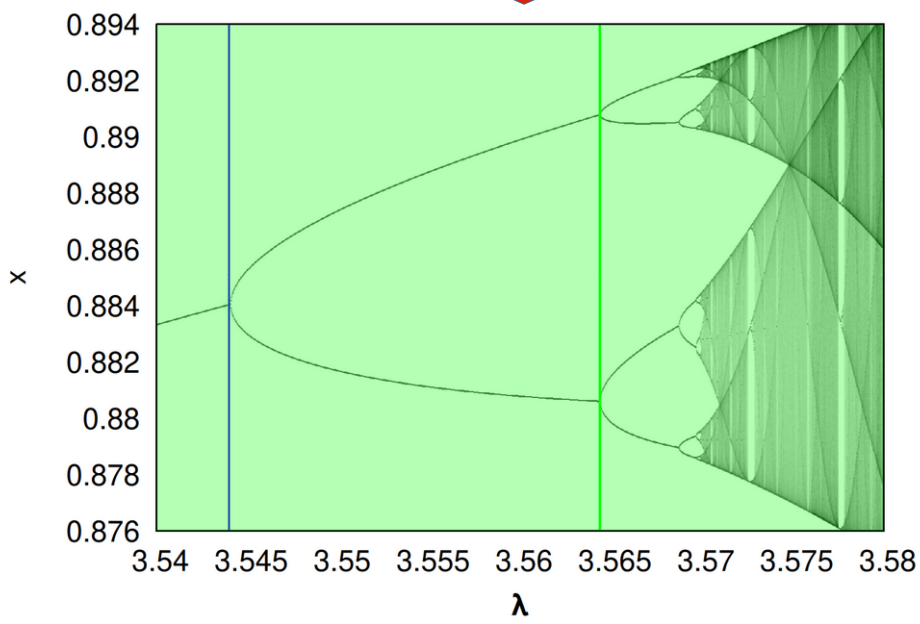
bifurcation diagram for the logistic map



self-similarity: fractals







That's not all the suprises....

Let's compare the distances between two consecutive points at which the period is doubled

$$\Delta_n = \frac{\lambda_{n-1} - \lambda_{n-2}}{\lambda_n - \lambda_{n-1}}$$

n	$period$	λ	Δ_n
1	2	3	–
2	4	3.449489	–
3	8	3.54409	4.7505
4	16	3.564407	4.66056
5	32	3.568759	4.6684
6	64	3.569691	4.6686

$$\lim_{n\to\infty} \Delta_n = 4.669201$$ ← **Faigenbaum constant**

When the chaotic dynamics is preceded by the period-doubling,
then this doubling always is govern by Faigenbaum constant
- irrespective of the physics that drives the dynamics.

In other words, this feature is **universal** for this class of systems.

Henon map (2-dimensional iterative map)

one-dimensional map

$$x_{i+1} = f(x_i)$$

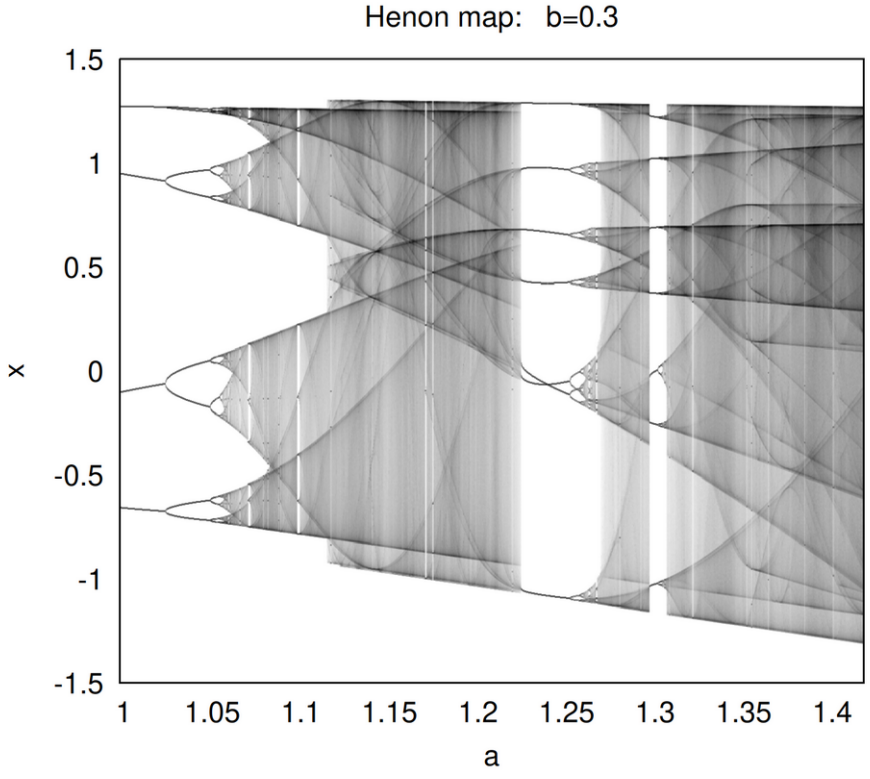


n-dimensional map

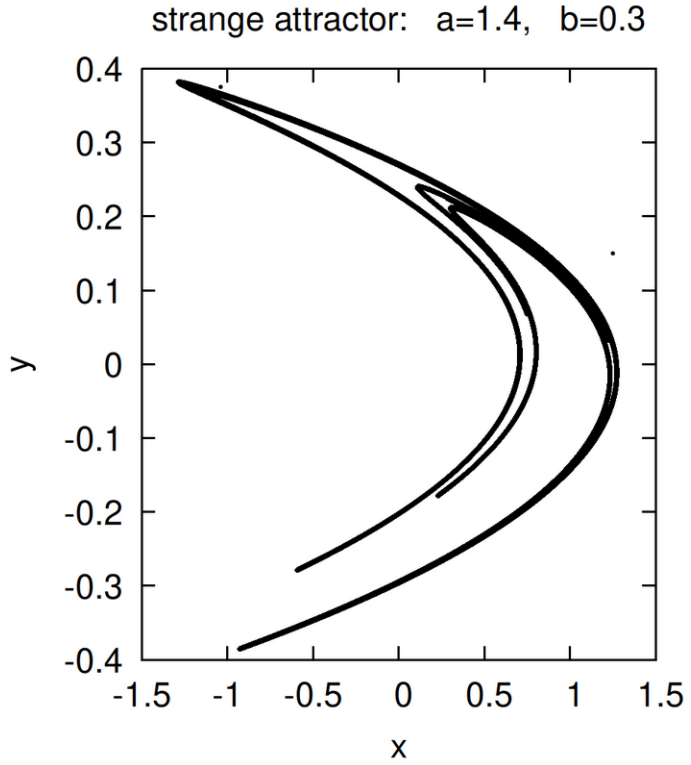
$$\vec{x}_{i+1} = \vec{f}(\vec{x}_i)$$

Simple example is a 2-dimensional map due to Henon:

$$x_{i+1} = 1 - ax_i^2 + y_i$$
$$y_{i+1} = bx_i$$



- **strange attractor exhibits self-similarity**



Chaotic motion of pendulum

Now we consider the physical system which may exhibit the chaotic motion:

periodically driven dumped pendulum

- equation of motion

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - b\dot{\theta} + F\cos(\Omega t)$$

$\omega^2 = \frac{g}{l} = 1$ - frequency of pendulum oscillations (b=F=0)

$b = 0.5$ - strength of friction

$\Omega = \frac{2}{3}$ - frequency of driving force

- numerical solution – velocity Verlet method:

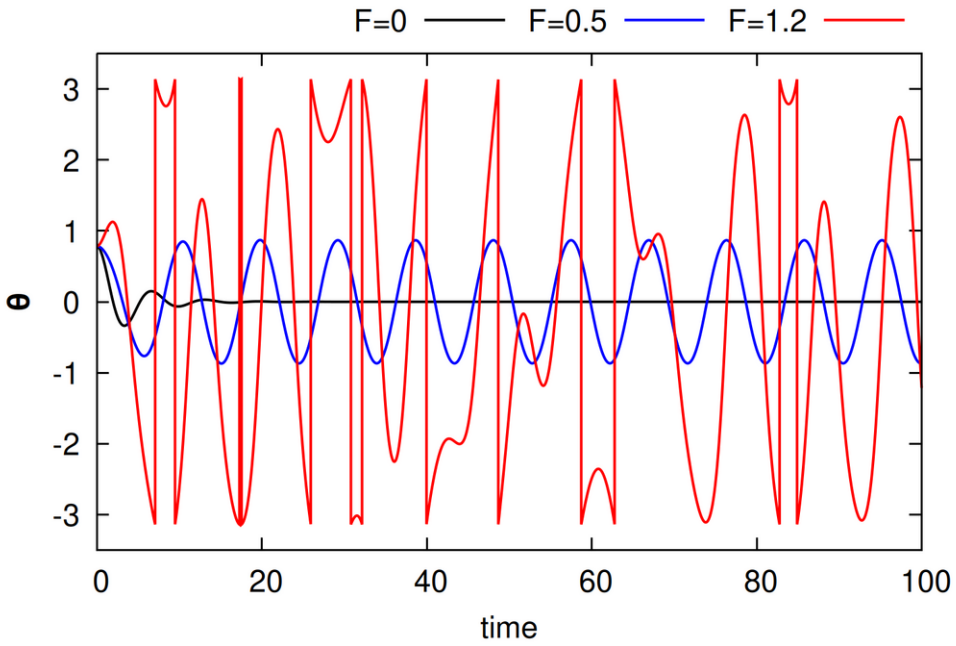
$$\dot{\theta}_{i+1/2} = \dot{\theta}_i + \frac{\Delta t}{2}\ddot{\theta}_i$$

angle/displacement: $\theta_{i+1} = \theta_i + \Delta t\,\dot{\theta}_{i+1/2}$

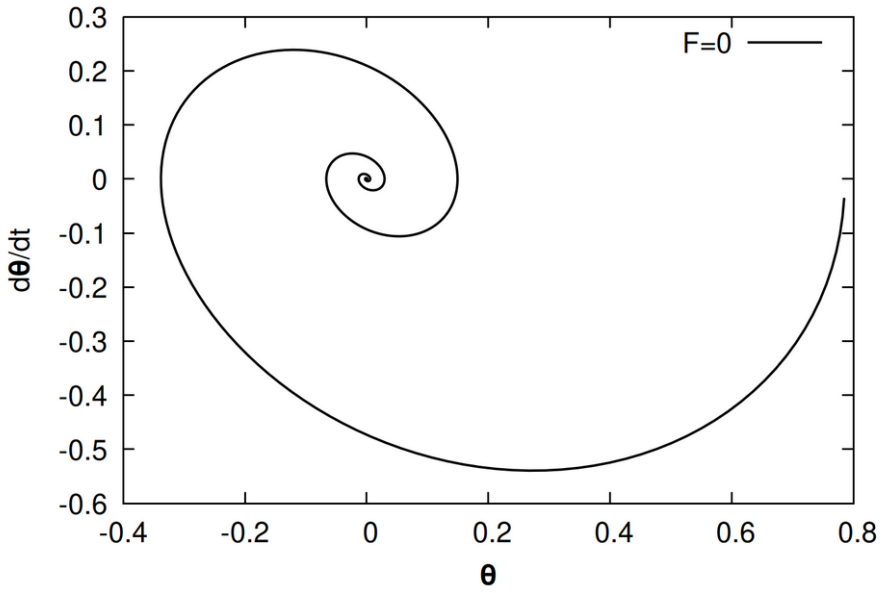
angular velocity: $\dot{\theta}_{i+1} = \frac{1}{1+\frac{\Delta t}{2}b}\left(\dot{\theta}_{i+1/2} + \frac{\Delta t}{2}\left[-\omega^2\sin\theta_{i+1} + F\cos(\Omega t_{i+1})\right]\right)$

this factor comes from shifting the velocity at t_{i+1} to the left side (see r.h.s. of EOM = acceleration) and then by dividing the equation by this denominator

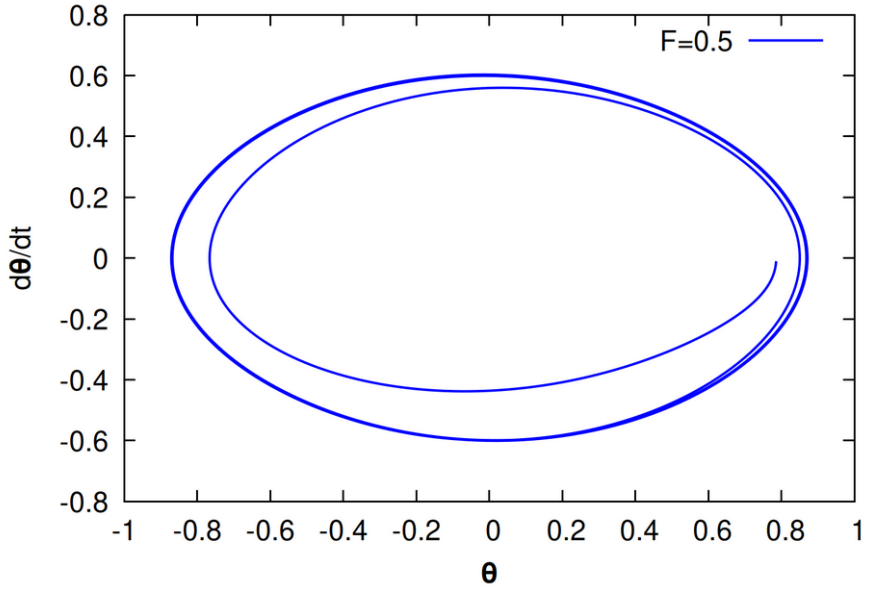
Phase diagrams



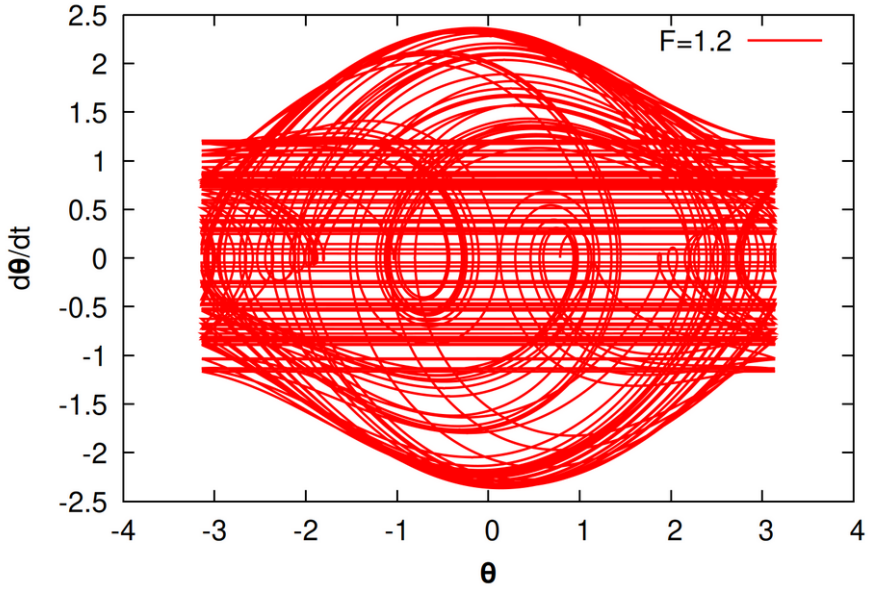
- trajectory spirals inward to a fixed point



- after transient begining, trajectory is settled at the attractor



- looks like a chaotic trajectory, but there are to many details in the plot

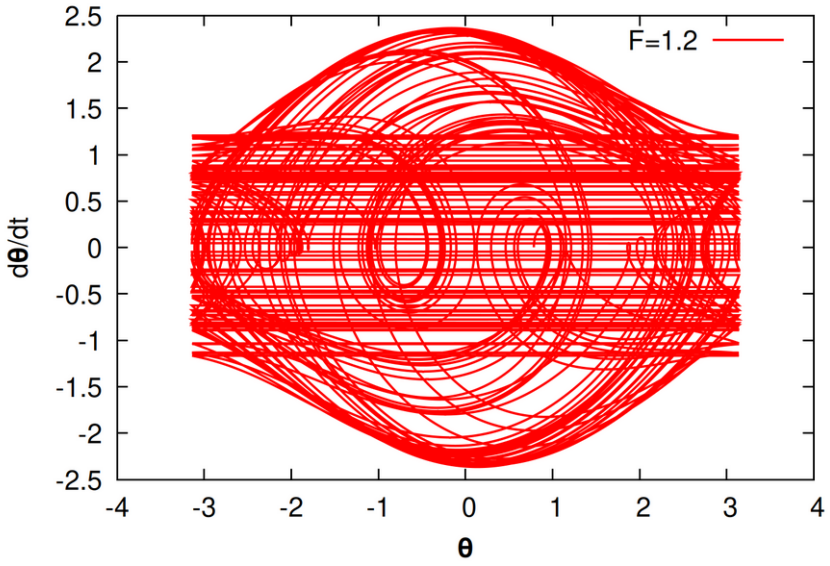


Poincare map

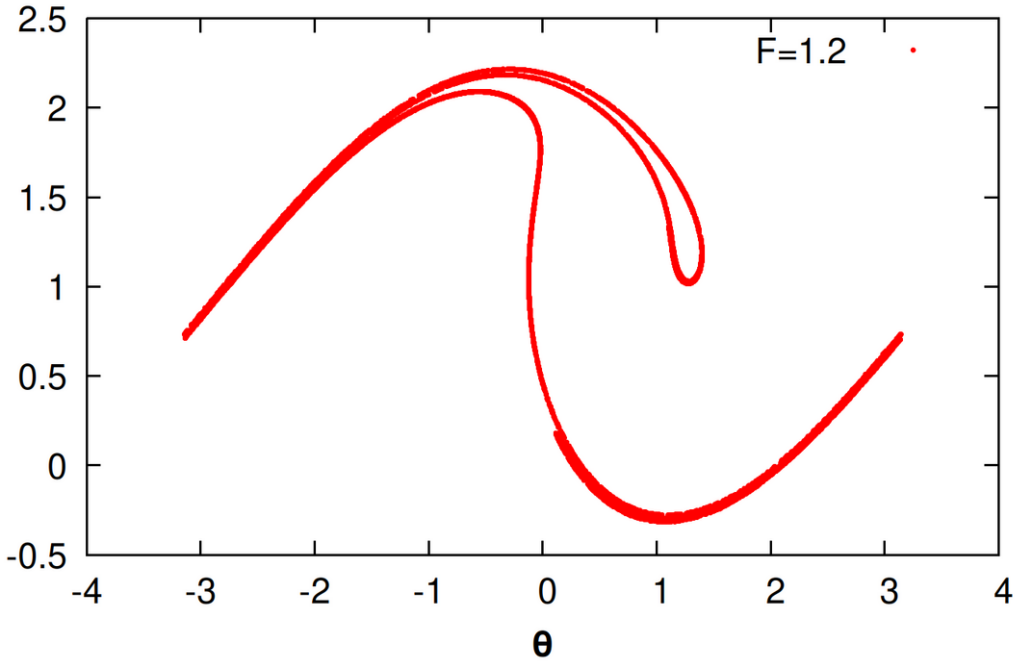
In order to limit the number of details in phase diagram for F=1.2 case this can be transformed into the Poincare map – there are recorded the data at the time instants which are multiple of driving force period

$$T_\Omega = \frac{2\pi}{\Omega} \quad \Rightarrow \quad t_n = T_\Omega,\ 2T_\Omega,\ 3T_\Omega,\ \ldots$$

ordinary phase diagram



Poincare map: $d\theta/dt=\theta\ (nT_\Omega)$



- the points recorded along the trajectory at time instants nT_Ω falls on an attractor, this in turn means that **the motion is chaotic but not random**
- dynamic is driven by a physical law and even though the motion is largely perturbed, we may recognize the fixed pattern in this motion (attractor)

Lyapunov exponent

- one of the major feature of chaos dynamics is its strong dependence on the initial conditions
- it turns out that that small perturbation in initial conditions maycause the system to evolve along a completely different trajectory
- by measuring of the difference between the positions of the object which moves along two trajectories (two parallel simulations are needed) we may distinct between:

(i) trajectories stay close to each other (**deterministic/regular dynamic**)
or
(ii) trajectories move away as system evolves in time (**chaotic dynamic**)

The distance between both trajectories can be approximated with the **Lyapunov exponent**

$$\Delta(t) = |\vec{r}_1(t) - \vec{r}_2(t)| \approx e^{\lambda t}\Delta(0)$$

← λ is one of the measures of the chaotic dynamics

Remarks:

- when the dynamic is govern by N 1-st order differential equations, then there are needed N+1 independent variables (N+ time)
- in (N+1)-rank system there are N+1 Lyapunov exponents, these can be calculated as the eigenvalues of Jacobiean matrix at given time instant

The chaotic system has at least one positive Laypunov exponent

Where come from the other Lyapunov exponents?
Let's analyze the stability of N-rank **autonomous** SODE around **the fixed point**

$$\frac{d\vec{r}}{dt} = \vec{f}(\vec{r}) \qquad \vec{r}(t) = [r_1(t), r_2(t), \ldots, r_N(t)]$$

$$\left.\frac{d\vec{r}}{dt}\right|_{\vec{r}=\vec{r}^0} = 0 \quad \Longrightarrow \quad \vec{f}(\vec{r}^0) = \vec{0} \qquad \vec{r}^0 - \text{ fixed point}$$

Now we add a small perturbation to the solution

$$\vec{r}(t) = \vec{r}^0 + \vec{\delta}(t), \qquad |\vec{\delta}(0)| << 1$$

$$\frac{d}{dt}\left(\vec{r}^0 + \vec{\delta}(t)\right) = \frac{d\vec{\delta}}{dt} = \vec{f}\left(\vec{r}^0 + \vec{\delta}(t)\right) \approx \underbrace{\vec{f}(\vec{r}^0)}_{=0} + L\vec{\delta} + O(\vec{\delta}^2)$$

Jacobiean matrix

$$L = [L_{i,j}] = \left[\left.\frac{\partial f_i}{\partial r_j}\right|_{\vec{r}=\vec{r}^0}\right]$$

$$\frac{d\vec{\delta}}{dt} = L\vec{\delta}$$

Diagonalization of L gives right vectors (v) and eigenvalues (λ),
the solution of an autonomous problem is known

$$L\vec{v}_i = \lambda_i \vec{v}_i \quad \Rightarrow \quad \vec{\delta}(t) = \sum_{k=1}^{N} c_k(t)\vec{v}_k \quad \Rightarrow \quad c_k(t) = c_k(0)e^{\lambda_k t}$$

Remarks:

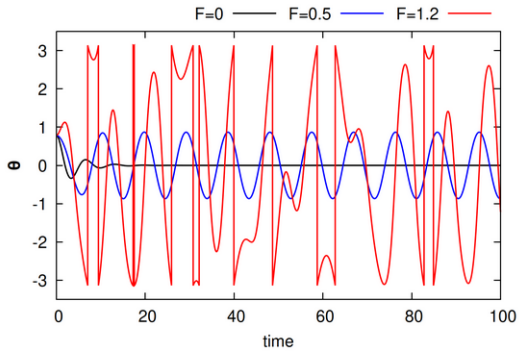
- positive eigenvalues of L may lead to exponentially divergent solutions when the initial conditions are slightly disturbed
- that is why the weather can be predicted only two weeks ahead (positive eigenvalues), what would happen next is unpredictable (we don't know all the initial conditions → perturbation)

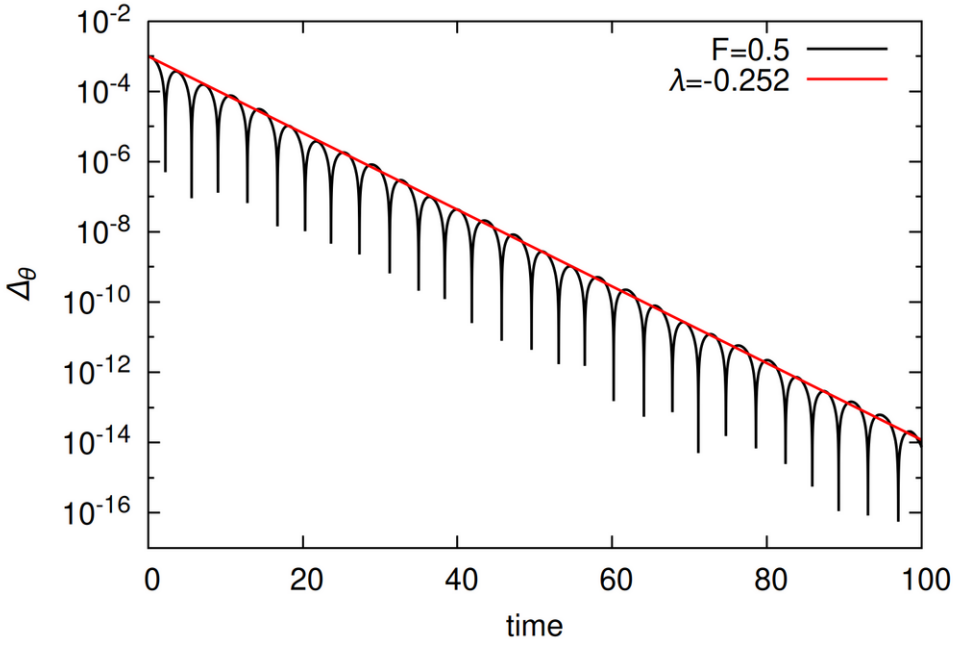
Simulation of deterministic chaos

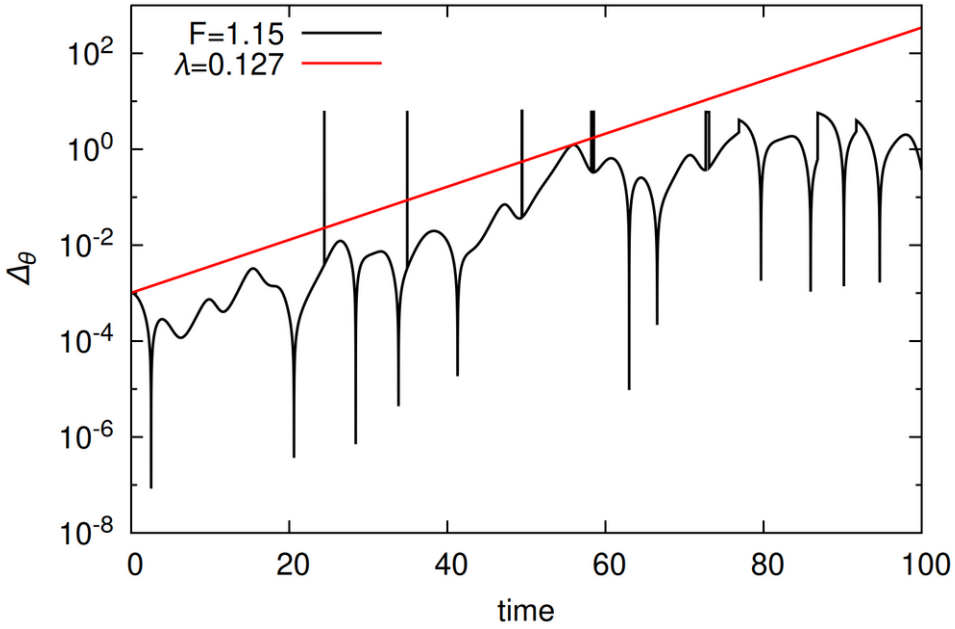
example: Lyapunov exponent for driven dumped pendulum

- we track the the trajectories of the pendulum which initial position was shifted by

$$\Delta(0) = 0.001$$







- dips are due to the crossing of trajectories, angle is artificially bound to a range

$$\theta \in [-\pi, \pi]$$

- trajectories stay close each other: $\lambda<0$

- initial distance between trajectories increases with time: $\lambda>0$
- maximum separation of trajectories is bound by 2π and cannot grow further

Lorenz equations

The coupled differential equations used for weather prediction
(simplified model of heating at bottom and cooling at the top)

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x(r - z) - y$$

$$\dot{z} = xy - bz$$

x,y,z – are the measures of the state of the system

x- fluid flow velocity around Rayleigh-Benard cell,
y - temperature difference between rising and falling fluid
z - difference in temperature between equilibrium temperature and the top as well as the bottom

σ and **b** are constant parameters
r – is varied

Fixed points are given for conditions

$$\dot{x} = \dot{y} = \dot{z} = 0$$

$$x_1 = y_1 = z_1 = 0$$

$$x_2 = -\sqrt{b(r-1)}, \quad y_2 = -\sqrt{b(r-1)}, \quad z_1 = r - 1$$

$$x_3 = -x_1, \quad y_3 = -y_1, \quad z_3 = z_1$$

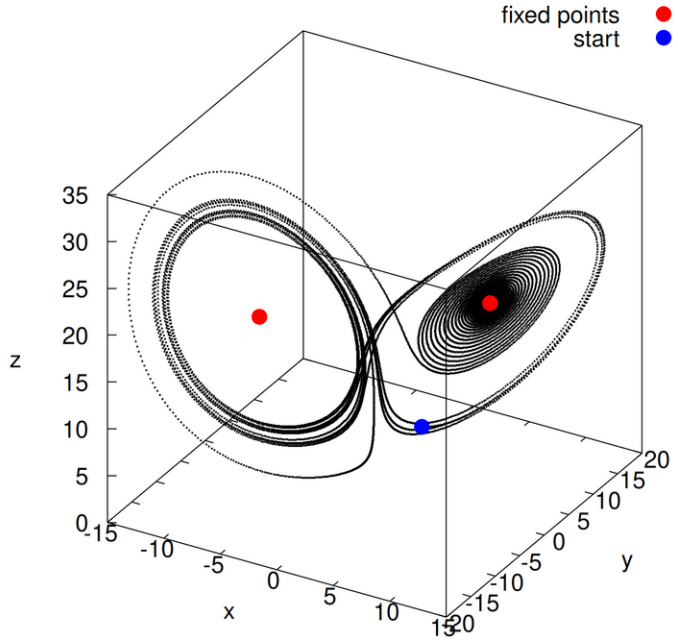
In simulation we take parameters:

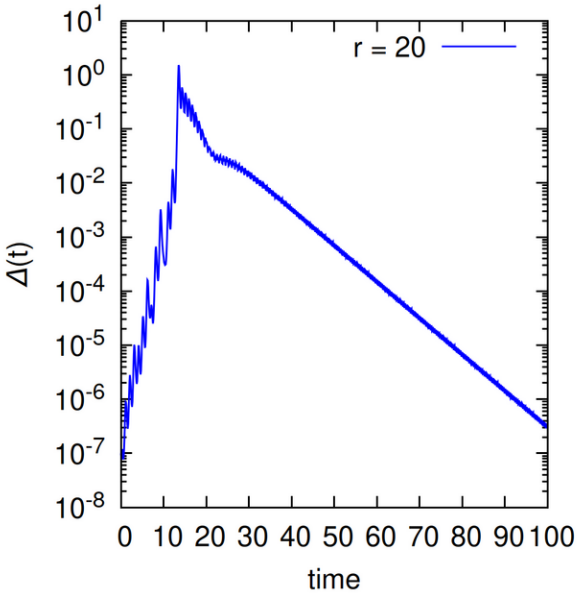
$$\sigma = 10, \quad b = \frac{8}{3}$$

- for r<24.7 solution settles at the 2-nd fixed point
- for r=28 system is chaotic

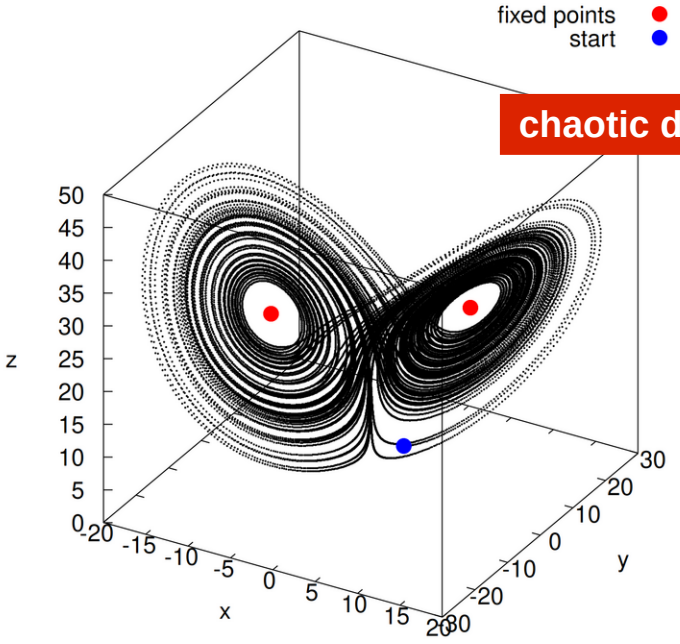
Simulation of deterministic chaos

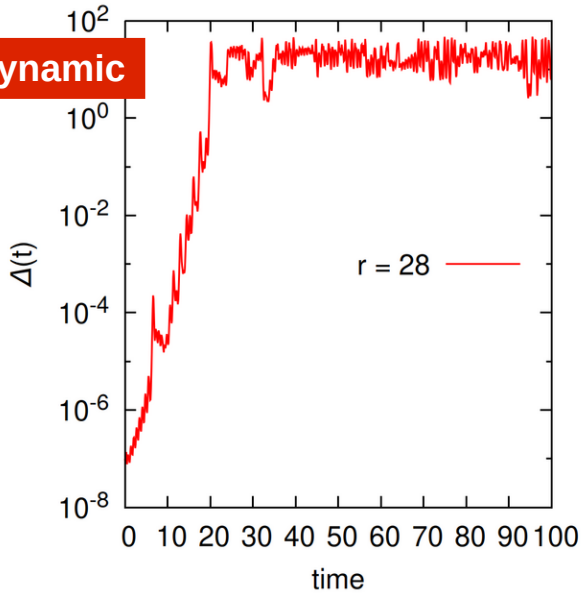
$$x(0) = 2, \quad y(0) = z(0) = 5, \quad \varepsilon = 10^{-7}, \quad \Delta(t) = |\vec{r}_1(t) - \vec{r}_2(t)|,$$





- in the initial transient state the trajectories seem to diverge
- then trajectories are getting close to each other and finally end at one of the fixed points





- solution jumps between two orbits
- each orbit is centered at one fixed point
- solution circulates around the fixed points but never fall on them

  (attracts and repels as well)