

Simulation of mechanical resonance for one-dimensional harmonic oscillator with RK4 method

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1 Introduction

Let's consider the one-dimensional motion of particle which is confined within parabolic potential $U(x) = kx^2/2$, in the course of its motion it slows down due to friction force F_α as well as may be driven by periodic external stimulus F_{ext} . These components of total force are defined as follow

$$F_{tot}(x, t) = F_u + F_\alpha + F_{ext} \quad (1)$$

$$F_u(x) = -\frac{dU}{dx} = -kx \quad (2)$$

$$F_\alpha(v, t) = -\alpha v = -\alpha \frac{dx}{dt} \quad (3)$$

$$F_{ext}(t) = F_0 \sin(\Omega_{ext}t) \quad (4)$$

where k is the force constant for returning force F_u , α is the friction parameter, F_0 and Ω_{ext} are the amplitude and the frequency of time periodic stimulus, respectively. Force exerted on body of mass m governs its dynamics, from second Newton's law we get immediately the equation of motion (EOM)

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x - \frac{\alpha}{m} \frac{dx}{dt} + \frac{F_0}{m} \sin(\Omega_{ext}t) \quad (5)$$

From the basic physics course we know there exist some analytical solutions for this EOM. If we consider vanishing periodic force $F_0 = 0$ then we get three regimes of damped motion. Denoting the characteristic frequency of undamped oscillator as $\omega_0 = \sqrt{k/m}$, the characteristic time scale due to friction $\tau = 2m/\alpha$ and x_0 and v_0 as initial position in space and the velocity, these solutions are following:

1. underdamped for $\omega_0\tau > 1$

$$x(t) = \sqrt{x_0^2 + \left(\frac{x_0 + v_0\tau}{\Omega_d\tau}\right)^2} e^{-\frac{t}{\tau}} \cos(\Omega_d t + \phi) \quad (6)$$

$$\phi = \arctan\left(\frac{x_0 + v_0\tau}{x_0\Omega_d\tau}\right) \quad (7)$$

$$\Omega_d = \sqrt{\omega_0^2 - \frac{1}{\tau^2}} \quad (8)$$

2. critically damped for $\omega_0\tau = 1$
3. overdamped for $\omega_0\tau < 1$

In this project we will focus on the first underdamped regime as it allows to observe the oscillations of decreasing amplitude, the other two describes monotonically (exponentially) fading amplitude without oscillations. The analytical solution will be used for checking the correctness of numerical data under dissipative conditions. Without dumping, the dissipation is absent and the total energy

$$E_{tot} = E_{kin} + E_{pot} \quad (9)$$

should be constant in the course of simulation, it has two components, the kinetic energy

$$E_{kin} = \frac{mv^2}{2} \quad (10)$$

and potential energy

$$E_{pot} = \frac{kx^2}{2} \quad (11)$$

By turning on the time periodic force $F_{ext}(t)$ the driven object will oscillate back and forth with the amplitude depending on the difference between frequency of periodic force Ω_{ext} and characteristic frequency of unperturbed harmonic oscillator ω_0

$$x_{max}(\omega) = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2 - \Omega_{ext}^2)^2 + \left(\frac{2\Omega_{ext}}{\tau}\right)^2}} \quad (12)$$

When the former is continuously adjusted to the same value of the latter the amplitude grows quickly indicating occurrence of mechanical resonance and only the dissipation of energy (second parameter in square root: $\tau = 2m/\alpha < \infty$) preserves the system from being physically destroyed.

2 Numerical algorithm

The EOM given in Eq.5 need to be solved with one of the methods devised for first-order ODE. In project we will use RK4 in its vector form. By first, 2-nd order ODE must be transformed into a system of 1-st order ODEs

$$\frac{dx}{dt} = v \quad (13)$$

$$\frac{dv}{dt} = \frac{F_{tot}}{m} \quad (14)$$

In the following we introduce more comfortable vector notation

$$\frac{d\vec{u}}{dt} = \vec{f}(t, \vec{u}) \quad (15)$$

$$\vec{u} = [x, v] = [u_0, u_1] \quad (16)$$

$$\vec{f}(t, \vec{u}) = \left[v, \frac{F_{tot}}{m} \right] = \left[u_1, -\frac{k}{m}u_0 - \frac{\alpha}{m}u_1 + F_0 \sin(\Omega_{ext}t) \right] \quad (17)$$

Equation 16 shows the sequence chosen for storing the dynamical variables in auxiliary vector $\vec{u}$, whereas Eq.17 is used in calculations of the right hand side of ODE system and for our convinience it shall be put in separate callable numerical procedure (function) which take $\vec{u}, k, m, \alpha, F_0, \Omega_{ext}, t$ as input parameters and returns $\vec{f}$. With this procedure at hand now we are in position to implement the RK4 method, thanks to the vector formulation of the differential problem this could take following general form

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define auxiliary vectors of size n (number of variables) and time variable:
     $\vec{w}, \vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4, \text{ time}$ 

calculate  $\vec{k}_1$ :
     $\vec{w} \leftarrow \vec{u}$ 
     $\text{time} \leftarrow t$ 
     $\text{calculate\_rhs\_ODE}(n, \vec{w}, k, m, \alpha, F_0, \Omega_{ext}, \text{time}, \vec{k}_1)$ 
calculate  $\vec{k}_2$ :
     $\vec{w} \leftarrow \vec{u} + \frac{\Delta t}{2} \vec{k}_1$ 
     $\text{time} \leftarrow t + \frac{\Delta t}{2}$ 
     $\text{calculate\_rhs\_ODE}(n, \vec{w}, k, m, \alpha, F_0, \Omega_{ext}, \text{time}, \vec{k}_2)$ 
calculate  $\vec{k}_3$ :
     $\vec{w} \leftarrow \vec{u} + \frac{\Delta t}{2} \vec{k}_2$ 
     $\text{time} \leftarrow t + \frac{\Delta t}{2}$ 
     $\text{calculate\_rhs\_ODE}(n, \vec{w}, k, m, \alpha, F_0, \Omega_{ext}, \text{time}, \vec{k}_3)$ 
calculate  $\vec{k}_4$ :
     $\vec{w} \leftarrow \vec{u} + \Delta t \vec{k}_3$ 
     $\text{time} \leftarrow t + \Delta t$ 
     $\text{calculate\_rhs\_ODE}(n, \vec{w}, k, m, \alpha, F_0, \Omega_{ext}, \text{time}, \vec{k}_4)$ 
update solution and time:
     $\vec{u} \leftarrow \vec{u} + \frac{\Delta t}{6} (\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4)$ 
     $t \leftarrow t + \Delta t$ 

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In this pseudocode we use previously defined function calculating the right hand side of 1-st order ode system, it is called four times returning adequate approximations $\vec{k}_i$ for intermediate time stages: t , $t + \Delta t/2$ and $t + \Delta t$. By enclosing pseudocode in separate procedure (function) it can be called iteratively within the loop, each call finds the solution for the next time step and shifts the time of simulation automatically.

3 Practical part

Tasks to do

1. Implement RK4 algorithm (Sec.2) on computer and write the function calculating the right hand side of system ODE (Eq.17). In simulation use following fixed values: $k = m = \omega_0 = 1$, initial position $x_0 = 1$ and velocity $v_0 = 0$, other parameters will change.
2. Testing correctness of the results.
 - (a) Set the parameters: $\alpha = 0$, $F_{ext} = 0$, $\Omega_{ext} = 0$, $t_{max} = 50$, $N = 10^4$, $\Delta t = t_{max}/N$. Perform simulation, and make one plot showing changes in E_{tot} , E_{kin} , E_{pot} . Draw trajectory $x(t)$ and variations $v(t)$ in time domain, make a plot of trajectory in phase space $x(v)$. Dissipation as well as the energy pumping is turned off, therefore the total energy is constant while the trajectory in phase space must be closed.
 - (b) Change the friction parameter to $\alpha = 0.1$ and repeat the simulation. Draw trajectory in time domain $x(t)$ and compare it with exact one (Eq.6). Make the plot of the trajectory in phase space $v(x)$. Plot the variations of three types of energies. When the energy dissipation is the fastest/slowest?

3. Energy dissipation.

Assume $F_{ext} = 0$, $t_{max} = 50$, $N = 10^4$ and perform simulations for $\alpha = 10^{-4}, 0.1, 0.5, 1.95$. Plot all trajectories $x(t)$ in one figure. The largest α value is close to critical one $\alpha_{crit} = 2$, what results should we expect for overdamped regime ($\alpha > 2$)?

4. Driving the damped system with external periodic force - mechanical resonance.

Assume $F_0 = 1$, $\alpha = 1.0$, $t_{max} = 10^3$, $N = 2 \cdot 10^5$. First make a test simulation for $\Omega_{ext} = 0.5\omega_0$. Plot trajectory $x(t)$, it should show some irregularity in amplitude and frequency at the beginning, then after some time the oscillations of object will synchronize with periodic stimulus giving stable amplitude. Now we are interested in reading off the amplitude of these driven oscillations. It is enough to find the height of the last maximum, for this purpose we need to track three subsequent solutions in time: the previous one x_{i-1} , present x_i and the next x_{i+1} . When the condition $x_i > x_{i-1}, x_{i+1}$ is fulfilled just save the instantaneous value of x_i , after many changes the last one will relates to the last detected maximum - this is the driven amplitude we are looking for. Perform simulations for $\Omega_{ext} = 0.1 \div 2.0$ with step $\Delta\Omega_{ext} = 0.01$ each time recording the amplitude of oscillations. Use this procedure for finding the dependence $x_{max}(\Omega_{ext})$ for $\alpha = 0.01, 0.1, 0.5, 1.0$. Plot these functions in one figure, add analytical result for one arbitrary chosen value of α for comparison. In this case we have the analytical solution at our disposal, but it happens very rarely and only the numerical solutions are then available.

5. At home prepare the report containing all generated results, comment on remarks addressed above.

4 Example results

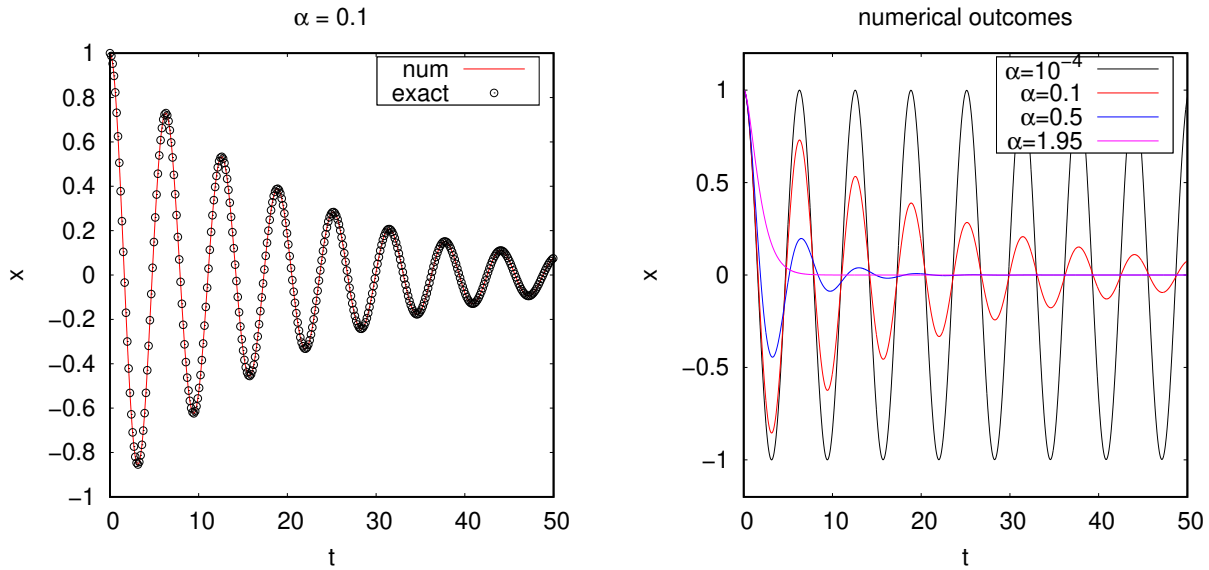


Figure 1: LEFT: Comparison of exact analytical solution with numerical ones for $\alpha = 0.1$, $F_{ext} = 0$. RIGHT: Numerical solutions $x(t)$ for increasing dumping parameter α , $F_{ext} = 0$.

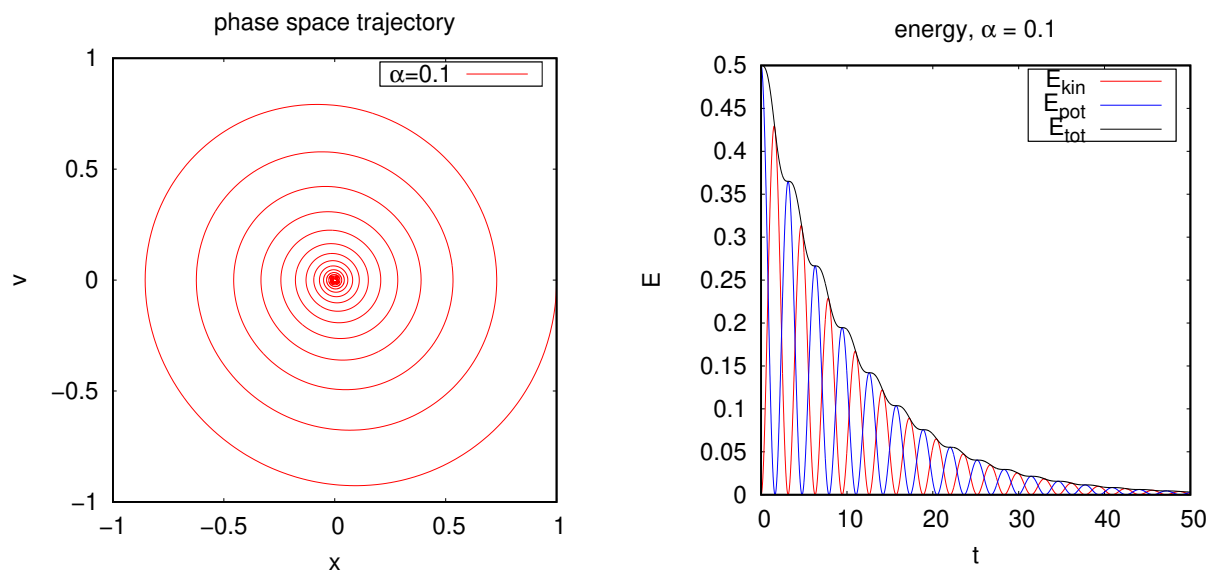


Figure 2: Phase space trajectory $v(x)$ (LEFT) and kinetic, potential and total energies (RIGHT) for dumping parameter $\alpha = 0.1$, $F_{\text{ext}} = 0$.

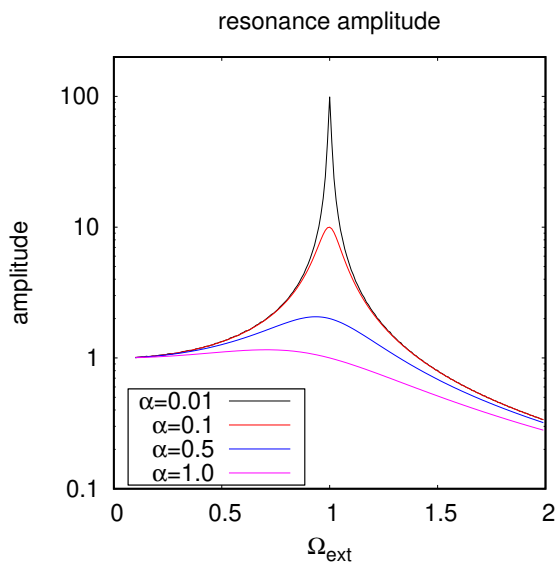


Figure 3: Amplitude of damped oscillator driven with periodic stimulus for few α values and $F_{\text{ext}} = 1$.